Project template: Continuously Changing Inflation Rate.

(The attached PDF file has better formatting.)

{Inflation may change continuously, not discretely. If the inflation rate changes from 5% in 20X1 to 12% in 20X8, we might model it as a continuous 1% *per annum* change.}

Jacob: How do we examine a continuously changing inflation rate?

Rachel: Use a $ß_2$ parameter for the starting inflation rate and a second parameter, such as $ß_{2B}$, for the annual change in the inflation rate. The inflation rate varies by calendar year.

Illustration: If the calendar years run from 0 to 14 and the inflation rate runs from 25% to 15%, code the inflation rate in the cell formulas as 25% – CalYr × 0.714%

- For CalYr = 0, the inflation rate is 25%.
- For CalYr = 14, the inflation rate is 25% – 14 × 0.714% = 15.00%.

Jacob: With this formula, what is the appropriate regression equation?

Rachel: We need the cumulative inflation from inception to the calendar year.

Illustration: If the inflation rate is 15% per annum, the total inflation from 2001 to 2010 is nine years of inflation. With continuous compounding, the total inflation is 9 × 15% = 135% $[(e^{15\%})^9 = e^{9 \times 15\%}]$.

With the declining inflation rate starting at 25% and decreasing 0.714% a year, the total inflation from 2001 to 2010 is

$$9 \times 25\% - \tfrac{1}{2} \times 9 \times 10 \times 0.714\%.$$

The proper regression equation has three independent variables: the development period, the calendar year, and the square of the calendar year. This type of regression is common, since many variables change smoothly.

Jacob: On the later SOA and CAS exams, we use duration and convexity for bond values, and deltas and gammas for option values. Is this related?

Rachel: Yes; the general formula for bond values and option values is

$$t \times \alpha + \tfrac{1}{2} \times t^2 \times \beta.$$

We use the same type of formula in the regression equation for smoothly changing values.

Jacob: This formula for bond prices and option values uses t^2; the formula for the changing inflation rate uses t × (t+1). What causes this difference?

Rachel: The formula here has t × (t+1) because the calendar years are discrete. Think of this as t × (t+k), where k is the time interval. For annual periods, k = 1. As the periods become small, $k \rightarrow 0$. As the change in the inflation rate becomes continuous, the formula becomes t^2.

Jacob: Is this still linear regression? One of the independent variables is $(X_2)^2$, which is the square of another variable.

Rachel: The regression is still linear in the parameters. We now have three independent variables: we use both the calendar year and the square of the calendar year. We don't care how the explanatory variables are defined. The relation between two explanatory variables causes multicollinearity, but the regression is still linear.

Jacob: What is the next step in the student project?

Rachel: Run the *REGRESSION* add-in using the three explanatory variables. Arithmetic errors are common, so check your results at each step.

Request residual output from the *REGRESSION* add-in, including residual plots. Format residual plots showing the shape of the line connecting the average residual.

The line in your residual plot should be a convex or concave parabola (∪ or ∩), when the regression uses only two explanatory variables. The shape and convexity of the parabola depends on the direction (increasing or decreasing) and magnitude of the change in the parameter.

With three explanatory variables that fully explain the change in the parameter, the line in the residual plot should be flat at the horizontal axis.

Stochasticity distorts the residual plots. With high stochasticity and few data points, the residual plots are hard to interpret. Start with a large change, such as from 35% to 5% over the 15 year period, with low stochasticity. Once you are sure you are doing the analysis correctly, use a more moderate change with higher stochasticity.

Jacob: I used an inflation rate that decreases from 35% to 5% over the 15 year period. The average inflation rate is ½ × (35% + 5%) = 20%. But the ordinary least squares estimator is not 20%.

Rachel: The data points are not evenly distributed over calendar years 0 to 14.

- 15 data points are in calendar year 0.
- 14 data points are in calendar year 1.
- …

- 1 data point is in calendar year 14.

Jacob: Another difference also affects the ordinary least squares estimators. Suppose the geometric decay (development period trend) is zero. The expected Y value is $\alpha + \beta_2 \times X_2$: the starting value plus cumulative inflation.

- The 15 data points in CY 0 are not affected by the inflation rate.
- The 14 data points in CY 1 are affected by inflation in CY 0 (one year of inflation).
- The 13 data points in CY 2 are affected by inflation in CY's 0 & 1 (2 years of inflation).

The different calendar years have different numbers of years of inflation.

Rachel: We think of inflation as moving forward in time. The expected value in year X+1 is the expected value in Year X times the inflation rate.

The regression equation is symmetric with respect to time. We can think of the intercept coefficient of the regression equation as the expected Y value in calendar year 14. The expected value in Year X is the expected value in Year X+1 divided by the inflation rate.

- The 1 data points in CY 14 is affected by the inflation rate.
- The 2 data points in CY 13 are affected by inflation in CY 14 (one year of inflation).
- The 3 data points in CY 12 are affected by inflation in CY's 13 & 14 (2 yrs of inflation).

Jacob: If the inflation rate changes continuously, what is the revised regression equation?

Rachel: We use $(X_2)^2$ as the independent variable for the change in the inflation rate.

- If inflation starts at zero and increases linearly, we use $(X_2)^2$ alone.
- If the inflation rate starts at positive figure and changes linearly, we use a combination of X_2 and $(X_2)^2$.

Illustration: Suppose inflation is R% in year 0 and increases Z% a year.

- Constant inflation of R% accumulated for N years is N × R%.
- An increase of Z% a year accumulated for N years is ½ × N × (N+1) × Z%.
- If the increase begins in the second year, the accumulated increase after N years is ½ × N × (N – 1) × Z%. The first year is R%; the second year is R% + Z%.
- The combination of the constant inflation and the annual increase is

$$N \times R\% - \tfrac{1}{2} \times N \times Z\% + \tfrac{1}{2} \times N^2 \times Z\%$$

Illustration: The inflation rate changes from 5% in 20X0 to 14% in 20X9. The year index is 0 for 20X0, 1 for 20X1, …, and 9 for 20X9.

- The constant inflation of 5% a year accumulated for N years gives cumulative inflation of 5% for 20X1, 10% for 20X2, and so forth. The 5% cumulative inflation in 20X1 is the inflation in 20X0.
- The change in the inflation rate is 1% in 0% in 20X0, 1% in 20X1, and so forth. These figures affect the cumulative inflation for the next calendar year.
- The cumulative *increase* in the inflation rate is 0% in 20X1, 1% in 20X2, 3% in 20X3, 6% in 20X4, 10% in 20X5, and so forth.

The formula gives

- 20X1: $1 \times 5\% - \frac{1}{2} \times 1 \times 1\% + \frac{1}{2} \times 1^2 \times 1\% = 5\%$
- 20X2: $2 \times 5\% - \frac{1}{2} \times 2 \times 1\% + \frac{1}{2} \times 2^2 \times 1\% = 11\%$
- 20X3: $3 \times 5\% - \frac{1}{2} \times 3 \times 1\% + \frac{1}{2} \times 3^2 \times 1\% = 18\%$

We translate these formulas into prose:

- The expected value in 20X1 is the expected value in 20X0 + 5%.
- The expected value in 20X2 is the expected value in 20X0 + 5% + 6%.
- The expected value in 20X3 is the expected value in 20X0 + 5% + 6% + 7%.

We add inflation (instead of multiplying) because we use logarithms of paid losses.

Year	*Inflation*	*Accumulated Inflation*	*Year Index*	*Index Squared*	*Formula*
20X1	5%	5%	1	1	5%
20X2	6%	11%	2	4	11%
20X3	7%	18%	3	9	18%
20X4	8%	26%	4	16	26%
20X5	9%	35%	5	25	35%
20X6	10%	45%	6	36	45%
20X7	11%	56%	7	49	56%
20X8	12%	68%	8	64	68%
20X9	13%	81%	9	81	81%

Take heed: The same formula appears on the financial economics sections of the actuarial exams. If interest rates change by Δ, the change in the market value of a bond is

$$-\Delta \times \text{Duration} + \frac{1}{2} \Delta^2 \times \text{Convexity}$$

If the stock price changes by Δ, the change in the value of a European call option is

$$\Delta \times \text{Delta} + \frac{1}{2} \Delta^2 \times \text{Gamma}$$

The $-\frac{1}{2} \times N \times Z\% + \frac{1}{2} \times N^2 \times Z\%$ in the formula above assumes discrete changes from year to year: 5% in all of 20X1, 6% in all of 20X2, and so forth. The bond and option formulas assume continuous changes.

Jacob: Must the change in the inflation rate be the same each year?

Rachel: We can derive regression equations for other patterns as well. A constant change each year makes the formulas simple and easy to verify.

Illustration: Your student project may assume the inflation rate decreases from 35% in year 0 to 5% in year 14, with a change of 2% each year. For your first simulation, use $\sigma = 0.01$, so that the stochasticity does not distort the expected values.

Write the inflation rate in calendar year X_2 as a function of the beginning rate and the annual change. Use discrete annual changes, such as 35% the first year, 33% the next year, 31% the next year, and so forth, for simplicity.

- The inflation rate in year X_2 is $35\% - 2\% \times X_2$.
- The total inflation from year 0 to year X_2 is $35\% \times X_2 - 2\% \times \frac{1}{2} \times X_2 \times (X_2 - 1)$.

Alternatively, assume inflation is an average of 34% in year 0, 32% in year 1, and so forth. Either assumption is fine.

Jacob: Does this regression equation have two or three explanatory variables?

Rachel: The simulation uses three independent variables: X_1, X_2, and $(X_2)^2$.

- X_2, and $(X_2)^2$ are correlated, so the variance of their least squares estimators is high.
- The correlation reduces the efficiency of the ordinary least squares estimators.
- It is harder to read the residual plots when the change in the slope is continuous.

Take heed: You might be tempted to say: “The inflation rate might change 5 percentage points over the 15 years, not 30 percentage points. The stochasticity σ for loss reserving is high; we should use $\sigma = 0.25$, not $\sigma = 0.01$.” These comments are correct, but the stochasticity overwhelms the inflation and the results are unclear.

Recommendation: If you want to examine a realistic scenario, add it at the end of your student project. Your write-up might say:

“The final scenario is a realistic portrayal of general liability loss reserves. The inflation rate changes from 10% in Year 0 to 5% in Year 14. The standard error σ is 0.25. The residual plot in my simulation is hard to interpret, since the expected pattern is distorted by the large year to year stochasticity.” You would explain the residual plots and the ordinary least squares estimators for the two explanatory variables.

Jacob: Do we re-examine the residual plots after adding the third independent variable?

Rachel: Yes. Verify that all three residual plots show horizontal lines. Copy the residual plots to your write-up and explain what they imply.

Take heed: This project template describes various scenarios. You can do the procedures outlined here, or you can substitute other items for your student project. You can choose a discrete change or a continuous change in a trend; you need not use both.

Jacob: What should we discuss for the stochasticity?

Rachel: Explain the effects of stochasticity and the number of observations on the results: the R^2, the residual plots, testing the stability of the parameters. You might examine:

- Using four times as many observations affects the R^2, the standard errors of the β parameters, and the standard errors of the forecasts. Test each of these by running regressions on a 15 by 15 matrix vs a 30 by 30 matrix.
- Using four times as many observations makes hypothesis testing more accurate. You might explain how to test whether the inflation rate is changing.

We do not prescribe specific elements of the student project. The homework assignments should be completed at specified in the modules. For the student project, we give guidance and show illustrations. You determine what you will examine and how you do it.