TS module 16 ARIMA forecasting with binary residuals

(The attached PDF file has better formatting.)

Standard ARMA processes assume a normal distribution for the residuals. This exercise uses a binary residual to clarify the variance of ARMA vs ARIMA processes.

Exercise 16.1: Binary Residuals: ARIMA vs ARMA

An MA(1) process has θ_1 = 1. The residual in each period is 1 or –1 with 50% chance of each. An ARIMA(0,1,1) process is the cumulative sum of the MA(1) process.

A. What are the variances of the 1, 2, & 3 period ahead forecasts for the ARMA process?
B. What are the variances of the 1, 2, & 3 period ahead forecasts for the ARIMA process?

Part A: The variance of the error term is $[1^2 + (-1)^2] / 2 = 1$. This is a population variance, so we divide by N, not by N-1.

- In a sample variance, the error terms are not necessarily +1 and –1.
- The population variance uses the distribution of error terms.

The one period ahead forecast for the ARMA process is $\mu + \epsilon_t - \theta_1 \times \epsilon_{t-1}$.

- The parameters μ and θ_1 are scalars with no variance.
- ϵ_{t-1} has already occurred, so it is also a scalar.
- ϵ_t has not yet occurred, so it is a random variable with a variance of 1.

⇛ The variance of $\mu + \epsilon_t - \theta_1 \times \epsilon_{t-1}$ is 1.

Part B: The one period ahead forecast for the ARIMA process is the one period ahead forecast for the ARMA process plus the most recent value of the ARIMA process.

- The most recent value of the ARIMA process has already occurred, so it is a scalar.
- The variance of the one period ahead forecast of the ARIMA process is the same as the variance of the one period ahead forecast of the ARMA process.

Part C: The two periods ahead forecast for the ARMA process is $\mu + \epsilon_{t+1} - \theta_1 \times \epsilon_t$.

- The parameters μ and θ_1 are scalars with no variance.
- ϵ_t and ϵ_{t+1} have not yet occurred.
- They are independent random variables with variances of 1.
- The variance of $\theta_1 \times \epsilon_{t-1}$ is $\theta_1^2 \times \text{var}(\epsilon_{t-1})$.
- The variance of the difference of two independent random variables is the sum of the variances of each random variable.

⇛ The variance of $\mu + \epsilon_t - \theta_1 \times \epsilon_{t-1}$ is $1 + \theta_1^2 \times 1 = 1 \times (1 + \theta_1^2) = 2$, since $\theta_1 = 1$.

Part D: The two periods ahead forecast for the ARMA process is the sum of

- The most recent value of the ARIMA process: Y_{t-1}
- The one period ahead forecast of the ARMA process.
- The two periods ahead forecast of the ARMA process.

$= Y_{t-1} + \mu + \epsilon_t - \theta_1 \times \epsilon_{t-1} + \mu + \epsilon_{t+1} - \theta_1 \times \epsilon_t$

$= [Y_{t-1} + 2 \times \mu - \theta_1 \times \epsilon_{t-1}] + [\epsilon_{t+1} + (1 - \theta_1) \times \epsilon_t]$

We have grouped the two periods ahead forecast into

- a sum of scalars (with no variance) and
- a sum of random variables.

$\theta_1 = 1$ so $(1 - \theta_1) = 0$, and the variance of the sum of random variables is $var(\epsilon_{t+1})$.

Intuition: Let $\mu = 0$ and $Y_{t-1} = 0$. The table below shows the possible values of the two periods ahead value of the ARMA process.

	Residuals		*Values*		
Scenario	ϵ_t	ϵ_{t+1}	*ARMA(t)*	*ARMA(t+1)*	*ARIMA(t+1)*
1	-1	-1	-1	0	-1
2	-1	1	-1	2	1
3	1	-1	1	-2	-1
4	1	1	1	0	1

For the ARMA process in Period t+1:

- The mean value is $(2 + 0 + 0 + -2) / 4 = 0$.
- The variance is $(2^2 + 0^2 + 0^2 + (-2)^2) / 4 = 8 / 4 = 2$.

For the ARIMA process in Period t+1:

- The mean value is $(-1 + 1 + -1 + 1) / 4 = 0$.
- The variance is $((-1)^2 + 1^2 + (-1)^2 + 1^2) / 4 = 4 / 4 = 1$.