TS Module 9 Non-stationary IMA processes.

(The attached PDF file has better formatting.)

Background: Starting points for non-stationary time series

Cryer and Chan use non-stationary processes that have a fixed starting point. If the non-stationary time series has no starting point, its mean and variance are not defined.

Illustration: Infinite ARIMA(0,1,1) process

Suppose Y_t is an ARIMA(0,1,1) process = an IMA(1,1) process:

$$W_t = \Delta Y_t = Y_t - Y_{t-1} = e_t - \theta e_{t-1} \quad Y_t = Y_{t-1} + e_t - \theta e_{t-1}$$ or

If the time series has no starting point, then $Y_t = \epsilon_t + (1 - \theta)\, \epsilon_{t-1} + (1 - \theta)\, \epsilon_{t-2} + \ldots$

Y_t has no mean and a non-finite variance. Each entry in the non-stationary time series is the sum of an infinite number of random variables that do not die out.

Jacob: The expected value of each ϵ_t is zero. Doesn't this imply that the mean of Y_t is zero?

Rachel: If the current value of Y_t is 1, the expected values of all future observations is also 1. If the current value of Y_t is –1, the expected values of all future observations is also –1. If a time series has a mean, then the expected value as $t \to \infty$ is this mean. In this time series, the expected value in the limit depends on the current value, so the process is not stationary.

This is a simplified explanation. A more rigorous analysis shows that the process has no mean. Suppose one asks: "What is the mean of the real number line?" One might say: "Start at zero. The probability of a value k is the same as the probability of the value –k, so the two values offset each other ⇛ the mean of the real number line is zero." This reasoning is not correct. One could also start at any point m and argue that the points m + k and m – k offset each other ⇛ the mean of the real number line is m. The correct answer it that the real number line has no mean. Similarly, a random walk has no mean.

To examine the pattern of non-stationary ARIMA processes, assume they begin at some time t = –m, so

$\Rrightarrow Y_t = \epsilon_t + (1 - \theta)\, \epsilon_{t-1} + (1 - \theta)\, \epsilon_{t-2} + \ldots + (1 - \theta)\, \epsilon_{-m} - \theta\, \epsilon_{-m-1}$

This is not really a restrictive condition. Most commonly, we forecast future values of a time series based on historical observations. Before the first observation, we assume all values of the time series are zero. We model the evolution of the mean and variance of the process.

**Question 1.2: ARIMA(0,1,1) process

Suppose $Y_t = Y_{t-1} + e_t - \theta e_{t-1}$

with $\theta = 0.4$ and $\sigma^2 = 4$ for $t > 0$ and $Y_t = 0$ for $t \leq 0$.

What is the variance of Y_2?

A. 5.440
B. 6.080
C. 7.520
D. 8.160
E. 20.320

Answer 1.2: B

Write the ARIMA(0,1,1) process in the long form and determine the variance of each random variable.

- $Y_2 = Y_1 + \epsilon_2 - 0.4\ \epsilon_1 = \epsilon_2 + (1 - 0.4)\ \epsilon_1 - 0.4\ \epsilon_0$
- The variance of Y_2 is $4 + (1 - 0.4)^2 \times 4 + 0.4^2 \times 4 = 6.080$

Jacob: If we don't specify that $Y_t = 0$ for $t < 0$, what is the variance of Y_t?

Rachel: The process is not stationary and has no variance.

Jacob: If we know Y_1 and forcast Y_2, what is the variance of the forecast of Y_2?

Rachel: The one period ahead forecast has only one random error term, so the variance is 4.

See Cryer and Chan, P94: equation 5.2.7, for IMA(1,1) process:

$$Var(Y_t) \quad \left[1 + \theta^2 + (1 - \theta)^2 (t + m)\right] \sigma_e^2$$

Jacob: Do we always assume the time series starts at time 0?

Rachel: Cryer and Chan assume it starts at Period –m. Final exam problems often assume it starts at Period 0 or Period 1. They can be solved by first principles, as in this exercise.

**Question 1.3: IMA(1,1) process

An IMA(1,1) process has $\theta = 0.4$ and $Y_t = 0$ for $t < -20$.

What is $\rho(Y_t, Y_{t-1})$ for $t = 50$?

A. –0.02
B. –0.05
C. +0.02
D. +0.05
E. +0.99

Answer 1.3: E

(See Cryer and Chan, chapter 5, page 94, equation 5.2.8)

Jacob: What is the intuition for this result?

Rachel: Each Y_t is the sum of t+20 random error terms. For t = 50, the correlation of Y_t and Y_{t-1} has 70 terms which overlap and one term which differs. The two elements are almost identical, whereas the range of possible values of Y_t is broad. As $N \to \infty$, the correlation of successive elements of a non-stationary IMA time series $\to 1$.

**Question 1.4: IMA(2,2) process

Which of the following is an IMA(2, 2) process?

A. $Y_t = 2\, Y_{t-1} - Y_{t-2} + e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$
B. $Y_t = Y_{t-1} - Y_{t-2} + e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$
C. $Y_t = Y_{t-1} - 2\, Y_{t-2} - e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$
D. $Y_t = Y_{t-1} - Y_{t-2} - e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$
E. $Y_t = Y_{t-1} + Y_{t-2} + e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$

Answer 1.4: A

(See Cryer and Chan chapter 5, page 94, equation 5.2.9)

IMA(2,2) process:

$Y_t = 2\, Y_{t-1} - Y_{t-2} + e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$

$\Rightarrow Y_t - Y_{t-1} = Y_{t-1} - Y_{t-2} + e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$

$\Rightarrow \nabla Y_t = \nabla Y_{t-1} + e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$

$\Rightarrow \nabla^2\, Y_t = e_t - \theta_1\, e_{t-1} - \theta_2\, e_{t-2}$

**Question 1.5: ARIMA processes

- A time series is zero for periods before $t = -m$.
- For $t > -m$, the value is $Y_t = \epsilon_t + (1 - \theta)\,\epsilon_{t-1} + (1 - \theta)\,\epsilon_{t-2} + \ldots + (1 - \theta)\,\epsilon_{-m} - \theta\,\epsilon_{-m-1}$

What type of ARIMA process is this time series?

A. ARIMA(0,1,1)
B. ARIMA(1,1,0)
C. ARIMA(1,1,1)
D. ARIMA(1,2,1)
E. ARIMA(2,2,1)

Answer 1.5: A

See equations 5.2.5 and 5.2.6 at the bottom of page **.

- Equation 5.2.5 is an IMA process: $Y_t = Y_{t-1} + \epsilon_t - \theta\,\epsilon_{t-1}$
- Equation 5.2.6 writes the IMA process in the form given above.

**Question 1.6: Drift of IMA(1,1) process

Let Y_t' be an IMA(1,1) process with a drift of zero.

Let $Y_t = Y_t' + \beta_0 + \beta_1 \times t$

- β_0 and β_1 are scalars, not random variables.
- t is the index for time.

What is the drift of Y_t?

A. β_0
B. β_1
C. $\beta_0 + \beta_1 \times t$
D. $1 + \beta_0$
E. $1 + \beta_1$

Answer 1.6: B

- $\beta_0 + \beta_1 \times t$ is a regression line with a slope (= drift) of β_1
- The drift of Y_t is the drift of $Y_t' + \beta_1 = \beta_1$

Intuition: The drift of Y_t is the mean of $\nabla Y_t = 0 + \beta_0 + \beta_1 \times t - (\beta_0 + \beta_1 \times (t\text{-}1)\ ) = \beta_1$

- Final exam problems may give a non-zero drift for Y_t' and values for β_0 and β_1.
- The drift of Y_t is the drift of $Y_t' + \beta_1$.