TS Module 7 ψ weights (filter representation)

(The attached PDF file has better formatting.)

Cryer and Chan use the term ψ weights; other statisticians use the term *filter representation*. The term filter representation in any practice problems means ψ weights.

We use ϕ parameters for autoregressive models and θ parameters for moving average models.

- The ϕ parameters relate future time series values to past time series values.
- The θ parameters relate future time series values to past residuals.

Moving average parameters have a finite memory, and autoregressive parameters have an infinite memory.

- For an MA(1) process, a random fluctuation in period T affects the time series value in period T+1 only.
- For an AR(1) process, a random fluctuation in period T affects the time series value in all future periods.

We can convert a ϕ parameter to an infinite series of θ parameters.

Illustration: A $\varphi_1 = 0.500$ is equivalent to an infinite series of θ parameters

$$\theta_1 = -0.500,\ \theta_2 = -0.250,\ \theta_3 = -0.125,\ \ldots \text{ where } \theta_j = -(0.500^j).$$

One might wonder: *Why convert a single parameter to an infinite series?*

Answer: Each θ parameter affects one future value. To estimate variances of forecasts, we convert autoregressive parameters into sets of moving average parameters. We call the new model a filter representation and represent the new parameters by ψ weights.

Take heed: The ψ parameters have the opposite sign of the θ parameters: $\theta = 0.450$ is $\psi = -0.450$. The model is the same, but the signs of the coefficients are reversed.

$$y_t = \delta + \varepsilon_t - \theta_1\,\varepsilon_{t-1} \text{ is the same as } y_t = \delta + \varepsilon_t + \psi_1\,\varepsilon_{t-1}$$

The general form of a filter representation is $y_t - \mu = \psi_0\,\varepsilon_t + \psi_1\,\varepsilon_{t-1} + \psi_2\,\varepsilon_{t-2} + \ldots$

- For a moving average model, $\mu = \theta_0$.
    - Cryer and Chan often use time series with a mean of zero.
    - For the values of other time series, add the mean.

See Cryer and Chan, chapter 4, page 55, equation 4.1.1.

Both moving average and autoregressive processes have filter representations.

- If the time series has only moving average parameters, $\psi_j = -\theta_j$.
- If the time series has autoregressive parameters, each ϕ_j is a series of ψ_j's.

The exercise below emphasizes the intuition. Once you master the intuition, the formulas are easy.

We examine the filter representation for autoregressive models and mixed models.

** Exercise 7.1: AR(1) ψ weights (filter representation)

An AR(1) model with $\varphi_1 = 0.6$ is converted to a filter representation

$$y_t - \mu = \psi_0\,\varepsilon_t + \psi_1\,\varepsilon_{t-1} + \psi_2\,\varepsilon_{t-2} + \ldots$$

A. What is ψ_0?
B. What is ψ_1?
C. What is ψ_2?
D. What is ψ_j?

Part A: ψ_0 is one for all ARIMA models. See Cryer and Chan, chapter 4, page 55.

Part B: If the *current error term* increases by 1 unit, the *current value* increases by one unit. The one period ahead forecast changes by $1 \times \varphi_1 = 1 \times 0.6 = 0.6$, so $\psi_1 = \phi_1$.

Part C: If the one period ahead forecast changes by $1 \times \phi_1 = 1 \times 0.6 = 0.6$, the two periods ahead forecast changes by $0.6 \times \phi_1 = 0.6^2$, so $\psi_2 = \phi_1^{\;2}$.

Part D: The same reasoning shows that $\psi_j = (\phi_1)^j$.

** Exercise 7.2: ARMA(1,1) ψ weights (filter representation)

An ARMA(1,1) model with $\varphi_1 = 0.6$, $\theta_1 = 0.4$ is converted to a filter representation $y_t - \mu = \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots$

Cryer and Chan drop the subscripts from parameters of AR(1), MA(1), and ARMA(1,1) processes, calling them ϕ and θ instead of ϕ_1 and θ_1.

A. What is ψ_0?
B. What is ψ_1?
C. What is ψ_2?
D. What is ψ_j?

Part A: ψ_0 is one for all ARIMA models.

Part B: Suppose the *current error term* increases by 1 unit.

- The moving average part of the ARMA process changes the forecast by $1 \times -\theta_1 = 1 \times -0.4 = -0.4$.
- If the *current error term* increases by one unit, the *current value* increases by one unit.
- The autoregressive part of the ARMA process changes the forecast by $1 \times \varphi_1 = 1 \times 0.6 = 0.6$.

The combined change in the forecast is $-0.4 + 0.6 = 0.2$. The change in the one period ahead forecast is $\varphi_1 - \theta_1$.

Take heed: The negative sign reflects the convention that moving average parameters are the negative of the moving average coefficients.

Part C: The one period ahead forecast increases 0.2 units (the result in Part B), so the two periods ahead forecast increases $0.2 \times \varphi_1 = 0.2 \times 0.6 = 0.12$ units.

Part D: Repeating the reasoning above gives $\psi_j = 0.6^{j-1} \times 0.2$.

** Exercise 7.3: ARMA(2,1) ψ weights (filter representation)

An ARMA(2,1) model with $\varphi_1 = 0.6$, $\varphi_2 = -0.3$, $\theta_1 = 0.4$ is converted to a filter representation $y_t - \mu = \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots$

A. What is ψ_0?
B. What is ψ_1?
C. What is ψ_2?
D. What is ψ_3?

Part A: ψ_0 is one for all ARIMA models.

Part B: Suppose the *current error term* increases by 1 unit.

- The moving average part of the ARMA process changes the forecast by $1 \times -\theta_1 = 1 \times -0.4 = -0.4$.
- If the *current error term* increases by one unit, the *current value* increases by one unit.
- The autoregressive part of the ARMA process changes the forecast by $1 \times \varphi_1 = 1 \times 0.6 = 0.6$.

The combined change in the forecast is $-0.4 + 0.6 = 0.2$. The change in the one period ahead forecast is $\varphi_1 - \theta_1$.

Part C: A 1 unit increase in the current error term increases the two periods ahead forecast two ways in this exercise:

- The one period ahead forecast increases 0.2 units (the result in Part A), so the two periods ahead forecast increases $0.2 \times \varphi_1 = 0.2 \times 0.6 = 0.12$ units.
- The *current value* increases 1 unit, so the φ_2 parameter causes the two periods ahead forecast to increase -0.3 units.

The change in the two periods ahead forecast is $0.12 - 0.3 = -0.18$ units, so $\psi_2 = -0.18$.

Take heed: The θ_1 parameter does not affect forecasts two or more periods ahead: an MA(1) process has a memory of one period. In contrast, an AR(1) process has an infinite memory. The φ_1 parameter affects all future forecasts.

Part D: If the number of periods ahead is greater than the maximum of *p* and *q* (2 and 1 in this exercise), the direct effects of the parameters is zero. We compute the combined effects: $\psi_3 = \varphi_1 \times \psi_2 + \varphi_2 \times \psi_1 = 0.6 \times -0.18 - 0.3 \times 0.2 = -0.168$.

** Exercise 7.4: AR(2) process ψ weights (filter representation)

An AR(2) model $y_t - \mu = \varphi_1 (y_{t-1} - \mu) + \varphi_2 (y_{t-2} - \mu) + \varepsilon_t$ has $\varphi_1 = 0.4$ and $\varphi_2 = -0.5$. We convert this model to an infinite moving average model, or the filter representation

$$y_t - \mu = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots$$

A. What is ψ_1?
B. What is ψ_2?
C. What is ψ_3?

Part A: Suppose the *residual* in Period *T* increases one unit. We examine the effect on the *value* in Period T+1.

- The *current value* increases 1 unit.
- The φ_1 coefficient causes next period's value to increase 0.4 units.

Part B: Suppose the *residual* in Period *T* increases one unit. We examine the effect on the *value* in Period T+2.

- The current value increases 1 unit.
- The φ_2 coefficient causes the two periods ahead value to increase –0.5 units.
- The φ_1 coefficient has a two step effect. It causes next period's value to increase 0.4 units and the value in the following period to increase 0.4 × 0.4 = 0.16 units.

The net change in the two periods ahead value is –0.5 + 0.16 = –0.34.

- The AR(2) formula is: $\psi_2 = \varphi_1^2 + \varphi_2 = 0.4^2 - 0.5 = -0.340$.
- The explanation above is the intuition for this formula.

Part C: We use all permutations: $\varphi_1 \times \varphi_1 \times \varphi_1$, $\varphi_1 \times \varphi_2$, and $\varphi_2 \times \varphi_1$ =

$$0.4^3 + 2 \times 0.4 \times -0.5 = -0.336$$

For this part of the exercise, the subscript of ψ is greater than the order of the ARMA process. Instead of working out all the permutations, we multiply each ϕ_j coefficient by the ψ_{k-j} coefficient. We multiply ϕ_1 by ψ_2 and ϕ_2 by ψ_1 = $0.4 \times -0.34 + -0.5 \times 0.4 = -0.336$

Take heed: The formulas are simple permutations.

- Focus on the intuition, not on memorizing formulas.
- The final exam problems can all be solved with first principles.