TS Module 7 Stationary mixed processes practice problems

(The attached PDF file has better formatting.)

*Time Series Practice Problems ARMA Means and Variances*

ARMA processes have autoregressive parameters $\phi_j$, moving average parameters $\theta_j$, a constant $\delta$ or $\theta_0$, and a standard error $\sigma^2$. We derive

- Means and variances of the $Y_t$ terms and an autocorrelation function.
- Forecasts with their standard errors and confidence intervals.

The practice problems here cover means and variances of the time series. Other sets of practice problems show how to work more complex items.

The textbook shows all the relations tested on the final exam and used in the student projects. As you review the modules, work through the derivations of each result. The relations follow directly from first principles.

The final exam problems emphasize the intuition. An exam problem might give the mean or variance of the time series and back into a coefficient. If you understand how autoregressive and moving average parameters affect the means, forecasts, variances, and standard errors, the problems are straight-forward.

*Question 7.1: Mean of ARMA Process

An ARMA(p,q) process is $Y_t = 3 + \epsilon_t$ + moving average parameters $\theta_j = 0.5^j$ for j = 1 to 4 and autoregressive parameters $\phi_k = 0.366^k$ for k = 1 to 2. Note that the constant term is 3. What is the mean of this time series?

A. 2
B. 4
C. 6
D. 8
E. 10

Answer 7.1: C

The constant term is called $\theta_0$ in the Cryer and Chan textbook. Other authors use $\delta$ or $\alpha$. The constant term causes a displacement, not a change in the time series pattern.

*Intuition:* If all observed terms are the mean and all residuals are zero, the forecast of the next term is the mean. Substitute $\mu$ for the Y terms and 0 for the $\epsilon$ terms, giving a linear equation in one unknown $\mu$.

For this exercise, $\mu = 3 + \mu \times (0.366^1 + 0.366^2) + 0 \times (\text{terms with } \theta)$

The mean $\mu$ of this process is $\mu = 3 / (1 - \sum\phi_j) = 3 / (1 - 0.366 - 0.366^2) = 5.999$

The textbook has two ways of writing an ARMA process:

- Using terms of $(Y_t - \mu)$, with no constant term.
- Using terms of $Y_t$, with a constant term.

They are equivalent.

*Question 7.2: Mean of ARMA Process

An ARMA(p,q) process is $Y_t = 3 + \epsilon_t$ + moving average parameters $\theta_j = 0.5^j$ for j = 1 to 4 and autoregressive parameters $\phi_k = 0.366^k$ for k = 1 to 2. All values for t < 10 were the mean, and all residuals were zero.

If the actual value $y_{10} = 3$, what is the *expected value* of $y_{11}$?

A. 3.0
B. 3.3
C. 3.6
D. 6.0
E. 6.4

Answer 7.2: E

- The residuals for $t < 10$ are zero (by the assumptions).
- The expected value for $t = 10$ is the mean; work out the residual for $t = 10$.

The moving average *coefficient* = the *negative* of the moving average *parameter*.

The mean $\mu$ of this ARMA process is $\mu = 3 / (1 - \sum\phi_j) \Rrightarrow$

$\mu = 3 / (1 - 0.366 - 0.366^2) = 5.999$

- The residual for $y_{10}$, or $\epsilon_{10}$, is $3 - 6 = -3$.
- The expected value of $y_{11} = 3 + 0.366 \times 3 + 0.366^2 \times 6 - 0.5 \times (-3) = 6.402$

*Question 7.3: Mean of ARMA Process

An ARMA(p,q) process has moving average parameters $\theta_j = 0.5^j$ for j = 1 to 4 and autoregressive parameters $\phi_k = 0.366^k$ for k = 1 to 2. The parameter $\delta = 5$. What is the mean of this time series?

A. 7
B. 8
C. 9
D. 10
E. 11

Answer 7.3: D

The formula relating $\mu$ and $\delta$ for an autoregressive process is $\mu = \delta / (1 - \sum\phi_j) \Rightarrow$
$\mu = 5 / (1 - 0.366 - 0.366^2) = 9.999$

To verify, we assume all past values are the mean of 10 and all past residuals are zero. The next value is expected to be $5 + 0.366 \times 10 + 0.366^2 \times 10 = 10.000$ .

*Question 7.4: Mean of ARMA Process

An ARMA(p,q) process has moving average parameters $\theta_j = 0.5^j$ for j = 1 to 4 and autoregressive parameters $\phi_k = 0.366^k$ for k = 1 to 2. The parameter $\delta = 5$. Assume that all values for t < 10 were the mean, and all residuals were zero.

If $y_{10} = 9$, what is the expected value of $y_{11}$? We worked out the mean in the previous problem, which gives the values for $y_9$ and $y_8$. The residuals for $t \leq 9$ are zero (by the assumptions). The expected value for t = 10 is the mean, from which we work out the residual for t = 10. The convention in the textbook is that the moving average coefficient is the negative of the moving average parameter.

A. 8
B. 9
C. 10
D. 11
E. 12

Answer 7.4: C

The residual for $y_{10}$, or $\epsilon_{10}$, is 9 – 10 = –1.
The expected value of $y_{11} = 5 + 0.366 \times 9 + 0.366^2 \times 10 - 0.5 \times (-1) = 10.005$

*Question 7.5: ARMA Mean

We use an ARMA(3,1) model to forecast a time series:

$$y_t = 0.3y_{t-1} + 0.2y_{t-2} + 0.2y_{t-3} + \delta + \varepsilon_t - 0.2\varepsilon_{t-1}$$

The *mean* of this time series is 9. What is the constant term δ of the time series?

A. –2.70
B. –2.25
C. –1.20
D. +2.25
E. +2.70

Answer 7.5: E

Cryer and Chan use the variable $\theta_0$ instead of $\delta$. The discussion forum postings use $\theta_0$, $\delta$, or $\alpha$. The final exam problems may derive the constant term, the mean, or one autoregressive parameter. The solution is straight-forward.

The mean is the constant term $\delta$ divided by $(1 - \sum\varphi_j)$. The autoregressive coefficients affect the mean; the moving average coefficients do not affect the mean.

$\mu = \delta / (1 - \sum\phi_j) \Rrightarrow \delta = \mu \times (1 - \sum\phi_j) = 9 \times (1 - 0.3 - 0.2 - 0.2) = 2.700$

*Question 7.6: AR(1) Process

We use a time series process to model a sequence of values:

$$y_t = -1.0 y_{t-1} + 6 + \varepsilon_t, \ with \ \sigma^2 = 4$$

What is the mean of this process?

A. 3
B. 6
C. –6
D. 12
E. The time series is not stationary and has no mean

Answer 7.6: E

If the absolute value of the $\varphi_1$ coefficient is more than or equal to one, the time series has no mean (is not stationary). For higher order processes, the conditions for stationarity are more complex.

*Jacob:* Why does the time series have no mean? Suppose $\phi_1$ is a negative number less than or equal to –1. Why can't we use the formula $\mu = \delta / (1 - \phi_1)$?

*Rachel:* Suppose $\phi_1$ is –2 and $y_1$ is zero. The formula $\mu = \delta / (1 - \phi_1) = 6 / (1 - -2) = 2$.

The expected values of the next entries are +6, –6, +18, –30, +66, and so forth. The expected values assume $\epsilon = 0$.

The values oscillate about 2, but they move away from 2; they don't approach 2.

*Jacob:* What if $\phi_1 = -1$, as in the practice problem here?

*Rachel:* The formula gives $\mu = \delta / (1 - \phi_1) = 6 / (1 - -1) = 3$.

The expected values of the next entries are +6, 0, +6, 0, +6, 0, and so forth.

The values oscillate about 3, but they don't converge or diverge. The series has no mean.

*Take heed:* The mean $\mu$ is the expected value as $t \to \infty$. Even if an error term $\epsilon$ moves the time series away from this mean for a few periods, later values come back to the mean. In this time series, if a random fluctuation moves the process away from 3, later terms do not revert to 3.

*Question 7.7: AR(1) Process

We use an time series process to model a sequence of values:

$$y_t = -1.0 y_{t-1} + 6 + \varepsilon_t, \ with \ \sigma^2 = 4$$

What is $\gamma_0$, the variance of $y_t$?

A. 4
B. 6
C. 8
D. 12
E. The time series is not stationary and has no finite variance.

Answer 7.7: E

The $\varphi_1$ coefficient is greater in absolute value than one. The uncertainty in the forecasts increases without bound as the number of periods ahead increases. The variance of $y_t$ is like the variance of the forecast for an infinite number of periods ahead.

*Jacob:* Is the variance of $Y_t$ the same as the variance of the error term?

*Rachel:* Distinguish the variance of the error term and the variance of the Y terms.

Suppose we know the past terms of the time series and we must estimate the next value.

- The value is stochastic, so we do not know it with certainty.
- We estimate the distribution of this value.
- The distribution is normal with a variance $\sigma^2$.

Suppose we do not know any past terms of the time series and we estimate the next value.

- The mean of the distribution depends on the past values.
  - For an AR(1) process, it depends on one past value.
- For other ARMA processes, it may depend on several past values.
  - Each unknown value adds uncertainty and increases the variance of the estimate.
  - We combine the various sources of uncertainty.

If the variances of the sources are independent, we add them. Some sources of uncertainty are correlated, such as autoregressive and moving average parameters for the same period.

In an ARMA(1,1) process, the uncertainty is perfectly correlated. We add the autoregressive and moving average coefficients (i.e., subtract the parameters).

Think of the variances in a continuum.

- The variance of the one period ahead forecast is the variance of the error term.
- The variance of the *l* period ahead forecast as $l \rightarrow \infty$ is the variance of $Y_t$.
- In between are the variance of other forecasts.

*Question 7.8: AR(1) Process

We use an AR(1) process (autoregressive of order 1) to model a time series:

$$y_t = 0.4 y_{t-1} + 12 + \varepsilon_t$$ , with $\sigma^2 = 21$.

What is $\gamma_0$, the variance of $y_t$?

A. 2.1
B. 7
C. 21
D. 25
E. 49

Answer 7.8: D

$21 / (1 – 0.4^2) = 25.000$

This is *not* the variance of the forecast, which assumes we know the past values. This is the variance of the time series value itself.

Each $Y_t$ is the sum of two random variables: $\varepsilon_t$ and $0.4y_{t-1}$. In an autoregressive process, $y_t$ depends on the lagged values, but the residuals are independent of the lagged values.

You can solve this problem by the formula in the textbook or by intuition. The intuition is that the variance is the same for all elements of the time series. The variance of $y_t$ equals the variance of $y_{t-1}$. Form an equation for the relation of the variances from the time series equation $\text{Var}(y_t) = 0.4^2 \times \text{Var}(y_{t-1}) + 0 + \sigma^2$

*Jacob:* In the regression analysis course, the variance of Y is the variance of the error term. Why is that not true here?

*Rachel:* In the regression analysis course, Y is a function of X, which is not a random variable. In an autoregressive time series, $Y_t$ is a function of $Y_{t-1}$, which is a random variable.

*Question 7.9: Moving Average Model of Period N

A time series follows a moving average model of period N:

$$y_t = 1 + \varepsilon_t + \frac{\varepsilon_{t-1}}{N} + \frac{\varepsilon_{t-2}}{N} + \ldots + \frac{\varepsilon_{t-N}}{N}, \ \text{with } \sigma_\varepsilon^2 = 1$$

What is the mean of this time series?

A. ½
B. 1
C. 1 + 1/N
D. 2
E. The time series is not stationary and does not have a mean

Answer 7.9: B

For a moving average process, $\mu = \delta = \theta_0 = 1$ (in this problem).

The mean does not depend on the moving average parameters.

*Question 7.10: Moving Average Model of Period N

A time series follows a moving average model of period N:

$$y_t = 1 + \varepsilon_t + \frac{\varepsilon_{t-1}}{N} + \frac{\varepsilon_{t-2}}{N} + \ldots + \frac{\varepsilon_{t-N}}{N}, \ with\ \sigma_\varepsilon^2 = 1$$

What is the variance of this time series?

A. 1
B. (N + 1)/N = 1 + 1/N
C. $(N + 1)^2/N^2$
D. 2
E. The time series is not stationary and does not have a variance

Answer 7.10: B

For a moving average process, the residuals are independent and have the same variance.

- The variance of the sum of the residuals is the sum of the variances.
- The variance of $\alpha$Y, where $\alpha$ is a scalar and Y is a random variable, is $\alpha^2$ × variance(Y).

A moving average process is stationary if the sum of the moving average parameters is finite.

The variance is $\sigma^2 \times (1 + N \times 1/N^2) = 1 \times (1 + 1/N) = 1 + 1/N$