TS Module 5 moving average MA(2) practice problems

(The attached PDF file has better formatting.)

Time series MA(2) process practice problems

** Exercise 5.1: Variances, autocovariances, and autocorrelations of moving average process of order 2

A moving average process of order 2 is $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$, with $\theta_1 = 0.7$, $\theta_2 = 0.5$, and $\sigma_e = 2$.

A. What is γ_0, the variance of Y_t?
B. What is γ_1, the autocovariance of Y_t and Y_{t-1}?
C. What is γ_2, the autocovariance of Y_t and Y_{t-2}?
D. What is ρ_1, the autocorrelation of Y_t and Y_{t-1}?
E. What is ρ_2, the autocorrelation of Y_t and Y_{t-2}?

Part A: $\gamma_0 = (1 + \theta^2_1 + \theta^2_2) \times \sigma^2_e = (1 + 0.49 + 0.25) \times 2^2 = 6.960$

Y_t is the sum of three independent random variables: e_t, e_{t-1}, and e_{t-2}, with coefficients of 1, $-\theta_1$, and $-\theta_2$.

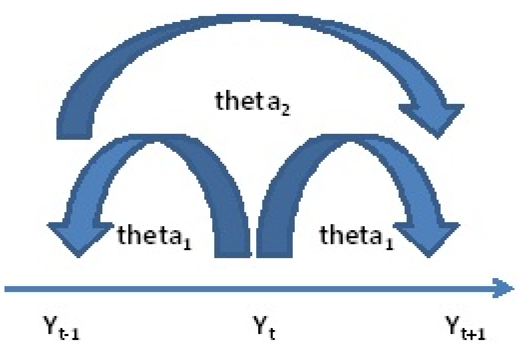
Each random variable has a variance of σ^2_ε. The variance of the sum is the sum of the variances of each term times the square of its coefficient.

Part B: $\gamma_1 = (-\theta_1 + \theta_1 \cdot \theta_2) \times \sigma^2_\varepsilon = (-0.7 + 0.7 \times 0.5) \times 2^2 = -1.400$

Jacob: γ_1 is the autocovariance of lag 1. Why does θ_2 affect this autocovariance? If the random variable ϵ_{t-1} increases one unit, Y_{t-1} increases one unit and Y_t increases $1 \times -\theta_1$ units. The covariance is in terms of the variances: for each 1 variance change of ϵ_{t-1}, Y_t decreases $\theta_1 \times \sigma^2_\varepsilon$. I understand that θ_2 affects the two period ahead value of the time series; why does it affect the one period ahead value?

Rachel: Your reasoning shows the effect of ϵ_{t-1} on Y_t. Your conclusion is correct: the covariance of ϵ_{t-1} and Y_t is $-\theta_1 \times \sigma^2_\varepsilon$.

- This exercise asks for the covariance of Y_{t-1} and Y_t.
- Y_{t-1} may change randomly for two reasons: random fluctuations of ϵ_{t-1} and random fluctuations of ϵ_{t-2}.
- A one unit random fluctuation in ϵ_{t-2} causes a change of $-\theta_1$ in Y_{t-1} and $-\theta_2$ in Y_t.
- The resulting autocorrelation of Y_{t-1} and Y_t is $-\theta_1 \times -\theta_2 = \theta_1 \times \theta_2$.



rho1 for MA(2) process

Jacob: That explanation makes sense, but now I don't understand the autocovariance for an AR(2) process.

- In an AR(2) process, a change in Y_{t-1} may reflect a random fluctuation in ϵ_{t-1} or a random fluctuation in ϵ_{t-2}.
- A random fluctuation of 1 unit in ϵ_{t-1} causes a one unit change in Y_{t-1}, which causes a change of ϕ_1 in Y_t.
- A random fluctuation of one unit in ϵ_{t-2} causes a change of ϕ_1 in Y_{t-1} and a change of ϕ_2 in Y_t.

By your logic, ρ_1 for an AR(2) process should be $\phi_1 + \phi_1 \times \phi_2$, instead of just ϕ_1.

Rachel: For an MA(2) process, θ_1 considers the random fluctuation in just ϵ_{t-1}. The random fluctuation in ϵ_{t-2} is picked up by θ_2. For an AR(2) process, θ_1 considers all changes in Y_{t-1}, whether they come from random fluctuations in ϵ_{t-1} or ϵ_{t-2} or any other error term. The ϕ_1 parameter relates all these changes to Y_t

Jacob: Your explanation makes sense, but now I don't understand the ρ_2 in an AR(2) process.

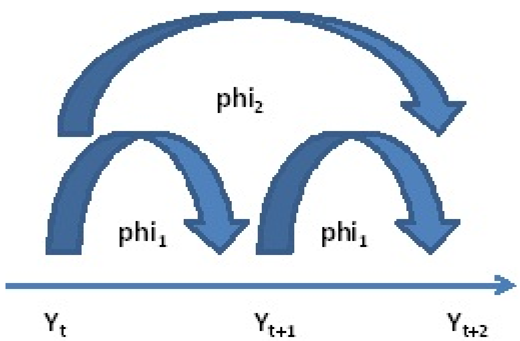
- ϕ_1 is the effect of Y_{t-1} on Y_t.
- ϕ_2 is the effect of Y_{t-1} on Y_{t+1}.

The covariance of lag 2, γ_2, is $(\phi_1^2 + \phi_2) \times \sigma^2_\varepsilon$. This seems to double count. The effect of Y_{t-1} on Y_{t+1} includes the two step effect of Y_{t-1} on Y_t and of Y_t on Y_{t+1}. Isn't this two step effect double counted in γ_2?

Rachel: You are confusing ϕ_2 with ρ_2.

- ρ_2 is the effect of Y_{t-1} on Y_{t+1}, including all multi-step effects.
- ϕ_2 is the *direct* effect of Y_{t-1} on Y_{t+1}, independent of any multi-step effects.

The formula for ρ_2 combines all the independent effects of the ϕ parameters.



rho$_2$ for AR(2) process

Part C: $\gamma_2 = (-\theta_2) \times \sigma^2 = -0.5 \times 2^2 = -2.000$

Part D: $\rho_1 = (-\theta_1 + \theta_1 \times \theta_2) / (1 + \theta_1^2 + \theta_2^2) = -0.201$

Part E: $\rho_2 = (-\theta_2) / (1 + \theta_1^2 + \theta_2^2) = -0.287$

(See Cryer and Chan page 62, equation at bottom of page)

For AR(1), AR(2), MA(1), MA(2), and ARMA(1,1) processes, know how to calculate γ_0, γ_1, γ_2, ρ_1, and ρ_2 from ϕ_1, ϕ_2, θ_1, and θ_2. For an MA(2) process:

$\gamma_0 = (1 + \theta_1^2 + \theta_2^2) \times \sigma^2$
$\gamma_1 = (-\theta_1 + \theta_1 \times \theta_2) \times \sigma^2$
$\gamma_2 = (-\theta_2) \times \sigma^2$

$\rho_1 = (-\theta_1 + \theta_1 \times \theta_2) / (1 + \theta_1^2 + \theta_2^2)$

$\rho_2 = (-\theta_2) / (1 + \theta_1^2 + \theta_2^2)$
$\rho_k = 0$ for $k = 3, 4, \ldots$

See Cryer and Chan, page 63 (equation 4.2.3)

**Exercise 5.2: Variance and autocorrelations of moving average process

A moving average process of order 2 is $Y_t = e_t - 0.6\ e_{t-1} - e_{t-2}$, with $\sigma^2_e = 1$

A. What is γ_0, the variance of Y_t?
B. What is γ_1, the autocovariance of lag 1?
C. What is γ_2, the autocovariance of lag 2?
D. What is ρ_1, the autocorrelation of lag 1?
E. What is ρ_2, the autocorrelation of lag 2?

Part A: $\gamma_0 = (1 + \theta^2{}_1 + \theta^2{}_2) \times \sigma^2{}_\varepsilon = (1 + 0.6^2 + 1^2) \times 1 = 2.36$

Intuition: Y_t is the sum of three independent random variables, with variances of 1, 0.36, and 1.

See Cryer and Chan, chapter 4, page 62, equation at the bottom of the page.

Part B: The autocovariance of lag 1 is $-\theta_1 \times (1 - \theta_2) = +0.6 \times (1 - 1) = 0$.

Part C: The autocovariance of lag 2 is $-\theta_2 = -1$.

Part D: See Cryer and Chan, chapter 4, page 63, equation 4.2.3.

$$\rho_1 = \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} \qquad \rho_1 = \frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$$

$(-0.6 + 0.6 \times 1) / (1 + 0.6^2 + 1^2) = 0/2.36$

ρ_1 is the correlation of $e_t - 0.6\ e_{t-1} - e_{t-2}$ with $e_{t-1} - 0.6\ e_{t-2} - e_{t-3}$

- The numerator has two non-zero terms: $-0.6\ e^2{}_{t-1}$ and $+0.6\ e^2{}_{t-2}$
- The error terms have the same variance, so these two terms cancel out.

Jacob: Does $\rho_2 = 0$ mean that the sample autocorrelation of lag 1 in this time series is zero?

Rachel: Actual time series are finite, so sample autocorrelations have random fluctuations. If the time series has 100 observations, the sample autocorrelation r_1 is distributed as a normal random variable with a mean of zero and a standard error of $1/\sqrt{100} = 0.10$. The observed autocorrelation is not zero. But if the time series is infinite, the sample autocorrelation of lag 2 approaches zero. This exercise assumes the moving average parameters are known and asks about the true autocorrelations.

Jacob: How should we think of this? Can one see the correlation intuitively?

Rachel: e_t is correlated with e_{t-1} two ways:

- Directly, with a correlation of $-\theta_1$.
- Indirectly through e_{t-2} and back to e_{t-1}, with a correlation of $-\theta_2 \times -\theta_1$.

Moving from e_{t-1} to e_{t-2} is like moving from e_{t-2} to e_{t-1}. We deal with these relations in more detail in modules 19 and 20 (seasonal time series), where $\rho_{11} = \rho_{13}$.

Part E: See Cryer and Chan, chapter 4, page 63, equation 4.2.3

$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$(-1) / (1 + 0.6^2 + 1^2) = -1/2.36 = -0.424$

Intuition: ρ_2 is the correlation of $e_t - 0.6\, e_{t-1} - e_{t-2}$ with $e_{t-2} - 0.6\, e_{t-3} - e_{t-4}$

- The numerator has one non-zero term: $-1 \times e^2_{t-2}$
- The denominator is the variance γ_0