

Corporate Finance Mod 21: Options, binomial tree pricing method, practice problems

**\*\* Exercise 21.1: Binomial tree pricing method**

- A stock is trading at \$100.
- Over the next six months, the stock price will increase 20% or decrease 20%.
- Six month European put and call options trade with exercise prices of \$110.
- The risk-free interest rate is 8% per annum compounded semi-annually, or 4% each six months.

- A. What is the risk-neutral probability that the stock price will increase over the next six months?
- B. What is the value of the call option if the stock price increases? If the stock price decreases?
- C. What is the delta of the call option?
- D. What is the replicating portfolio for the call option?
- E. What is the price of the call option?
- F. In a risk-neutral world, what is the expected value of the call option at the expiration date?
- G. What is the present value of the call option at inception?
- H. What is the value of the put option if the stock price increases? If the stock price decreases?
- I. What is the delta of the put option?
- J. What is the replicating portfolio for the put option?
- K. What is the price of the put option?
- L. In a risk-neutral world, what is the expected value of the put option at the expiration date?
- M. What is the present value of the put option at inception?
- N. Verify that the put and call values satisfy the put call parity relation.

*Part A:* The risk-neutral probability of an upward movement in the stock price is  $(r_f - d) / (u - d)$ , where  $r_f$  is the risk-free interest rate,  $u$  is the upward movement in the stock price, and  $d$  is the downward movement.

$$p = \frac{r_f - d}{u - d}$$

$$= (0.04 - (-0.20)) / (0.20 - (-0.20)) = 60\%.$$

*Part B:* If the stock price increases to  $\$100 \times 1.20 = \$120$ , the call option is worth  $\$120 - \$110 = \$10$ .

If the stock price decreases to  $\$100 \times 0.80 = \$80$ , the call option is worth zero.

*Part C:* The delta of the call option is the difference in price of the call option divided by the difference in price of the stock:  $(\$10 - \$0) / (\$120 - \$80) = 25\%$ .

*Part D:* The call option delta is 25%, so the replicating portfolio has  $25\% \times S$ . If the stock price increases to \$120,  $25\% \times S$  is \$30. The call option is worth \$10, so the replicating portfolio at the expiration date is  $25\% \times S - \$20$ , and the replicating portfolio now is the present value:  $25\% \times S - \$20 / 1.04 = 25\% \times S - \$19.23$ .  
*Verify:* if the stock price decreases to \$80,  $25\% \times S - \$20$  is  $\$20 - \$20 = \$0$ .

*Part E:* The price of the call option is  $25\% \times \$100 - \$20 / 1.04 = \$5.77$ .

*Part F:* In a risk-neutral world, the expected value of the call option at the expiration date is

$$60\% \times \$10 + (1 - 60\%) \times \$0 = \$6.$$

*Part G:* The present value of the call option at inception is  $\$6 / 1.04 = \$5.77$ .

*Part H:* If the stock price increases to  $\$100 \times 1.20 = \$120$ , the put option is worth zero.

If the stock price decreases to  $\$100 \times 0.80 = \$80$ , the put option is worth  $\$110 - \$80 = \$30$ .

*Part I:* The delta of the put option is the difference in price of the put option divided by the difference in price of the stock:  $(\$0 - \$30) / (\$120 - \$80) = -75\%$ .

*Part J:* The put option delta is  $-75\%$ , so the replicating portfolio has  $-75\% \times S$ . If the stock price increases to  $\$120$ ,  $-75\% \times S$  is  $-\$90$ . The call option is worth  $\$0$ , so the replicating portfolio at the expiration date is  $-75\% \times S + \$90$ , and the replicating portfolio now is the present value:  $-75\% \times S + \$90 / 1.04 = -75\% \times S + \$86.54$ .  
*Verify:* if the stock price decreases to  $\$80$ ,  $-75\% \times S + \$90$  is  $-\$60 + \$90 = \$30$ .

*Part K:* The price of the put option is  $-75\% \times \$100 + \$90 / 1.04 = \$11.54$ .

*Part L:* In a risk-neutral world, the expected value of the put option at the expiration date is

$$60\% \times \$0 + (1 - 60\%) \times \$30 = \$12.$$

*Part M:* The present value of the put option at inception is  $\$12 / 1.04 = \$11.54$ .

*Part N:* The put call parity relation is  $c + PV(X) = p + S$

$$\$5.77 + \$110 / 1.04 = \$111.54 = \$11.54 + \$100$$

## \*\* Exercise 21.2: Up and Down Movements

Jacob and Rachel own six month European put and call options with strike prices of \$80 on different stocks.

- Both stocks now trade at \$80.
- The risk-free rate is 1% each month (or  $1.01^{12}$  for the year).

Jacob's stock moves up or down 10% each month, and Rachel's stock moves up or down 20% each month.

- A. Whose stock has higher volatility?
- B. Whose call option is worth more?
- C. Whose put option is worth more?

*Part A:* The volatility of Rachel's stock is higher than that of Jacob's stock. The future value of Rachel's stock has a more diffuse probability distribution: specifically, a lognormal distribution with a higher  $\sigma$ .

*Part B:* Rachel's call option is worth more. Higher volatility raises the value of the call option.

*Part C:* Rachel's put option is worth more. Higher volatility raises the value of the put option.

We examine this relation by intuition, binomial tree pricing method (binomial lattice), Black-Scholes formula.

*Intuition:* Rachel's stock price may increase a lot or decrease a lot. If it increases a lot, Rachel has a large gain from her call option; if it decreases a lot, she does not lose anything on her call option. From her put option, Rachel gains a lot if the stock price decreases a lot but doesn't lose if the stock price increases a lot.

Jacob's stock price may increase less or decrease less. If it increases, Jacob has a smaller gain from the call option; if it decreases, he does not lose anything on the call option. From the put option, Jacob gains less if the stock price decreases but doesn't lose if the stock price increases.

*Question:* Does the effect of volatility on the option price depend on the stock price and the strike price?

*Answer:* The principle is true regardless of the stock price and strike price: higher volatility means a higher option price. The exact effect depends on the stock price and strike price. The effect is greatest when the stock price equals the present value of the strike price: the option is at the money. The effect is least when the option is far in-the-money or far out-of-the-money.

*Binomial Lattice:* Working out a six period lattice with pencil and paper is cumbersome. To simplify, we examine a one month binomial lattice.

The present value of the European call option is

the value of the call option at the expiration date if the stock price moves up  
× the risk-neutral probability of moving up / the risk-free rate

- For Jacob, if the stock price moves up 10%, the call option is worth  $\$88 - \$80 = \$8$ . The risk-neutral probability of moving up is  $(1.01 - 0.90) / (1.10 - 0.90) = 55.00\%$ . The present value of the call option is  $\$8 \times 55\% / 1.01 = \$4.36$ .
- For Rachel, if the stock price moves up 20%, the call option is worth  $\$96 - \$80 = \$16$ . The risk-neutral probability of moving up is  $(1.01 - 0.80) / (1.20 - 0.80) = 52.50\%$ . The present value of the call option is  $\$16 \times 52.5\% / 1.01 = \$8.32$ .

Rachel's call option is worth almost twice as much. If the stock price moves up, her call option pays twice as much, but the risk-neutral probability that her stock moves up is slightly lower.

*Black-Scholes formula:* A higher stock price volatility  $\sigma$  increases  $d_1$  and decreases  $d_2$ .

- For a call option, this increases  $S \times N(d_1)$  and decreases  $PV(X) \times N(d_2)$ , so it increases the present value of the call option.
- For a put option, this decreases  $S \times N(-d_1)$  and increases  $PV(X) \times N(-d_2)$ , so it increases the present value of the put option.

*Question:* The volatility is the total standard deviation of the stock price. The expected return of a stock depends on its systematic risk. If Rachel's stock has higher volatility and Jacob's stock has higher systematic risk, whose options are worth more?

*Answer:* The expected return of a stock depends on its systematic risk, not its total volatility. The value of an option depends on its total volatility, not its systematic risk. The principle:

For expected returns, look at systematic risk; for option prices, look at volatility (total risk).

**\*\* Exercise 21.3: Real vs risk-neutral probabilities**

A stock has a CAPM beta of one. The risk-free rate is 4% and the market risk premium is 8%. Over the next year, the stock price will either increase 20% with a probability of  $\frac{7}{8}$  (0.875) or decrease 20% with a probability of  $\frac{1}{8}$  (0.125).

- A. What is the expected return on the stock?
- B. What is the probability that the stock price will increase during the year?
- C. What is the risk-neutral probability that the stock price will increase during the year?

*Part A:* The expected return on the stock is  $4\% + 1.00 \times 8\% = 12\%$ .

*Part B:* Let the  $p$  be the probability that the stock price will increase.

$$12\% = p \times 20\% + (1 - p) \times -20\% \Rightarrow p = (12\% - (-20\%)) / (20\% - (-20\%)) = 80\%$$

*Part B:* Let the  $p$  be the risk-neutral probability that the stock price will increase.

$$4\% = p \times 20\% + (1 - p) \times -20\% \Rightarrow p = (4\% - (-20\%)) / (20\% - (-20\%)) = 60\%$$