TS module 20 seasonal models practice problems

** Exercise 20.1: Seasonal moving average process

A time series is generated by $Y_t = e_t - \Theta_1 e_{t-12} - \Theta_2 e_{t-24}$, with $\sigma^2_\varepsilon = \sigma^2$)

A. What is γ_0?
B. What is $\gamma_{12} = \gamma_{1 \times 12}$?
C. What is $\rho_{12} = \rho_{1 \times 12}$?

Part A: The variance of Y_t = variance $(e_t - \Theta_1 e_{t-12} - \Theta_2 e_{t-24}) = \sigma^2 + \Theta_1^2 \times \sigma^2 + \Theta_2^2 \times \sigma^2$.

Part B: The covariance of (Y_t, Y_{t-12}) = covariance $(e_t - \Theta_1 e_{t-12} - \Theta_2 e_{t-24}, e_{t-12} - \Theta_1 e_{t-24} - \Theta_2 e_{t-36})$ =

$$- \Theta_1 \times \sigma^2 + \Theta_1 \times \Theta_2 \times \sigma^2$$

Part C: The autocorrelation for one seasonal lag = $\rho_{12} = \rho_{1 \times 12}$ =

$(- \Theta_1 \times \sigma^2 + \Theta_1 \times \Theta_2 \times \sigma^2) / (\sigma^2 + \Theta_1^2 \times \sigma^2 + \Theta_2^2 \times \sigma^2) = (- \Theta_1 + \Theta_1 \times \Theta_2) / (1 + \Theta_1^2 + \Theta_2^2)$

** Exercise 20.2: Seasonal moving average process

A seasonal moving average process has a characteristic polynomial of $(1 - \theta x)(1 - \Theta x^{12})$.

For simplicity, assume $\sigma^2_e = 1$, so we can ignore the σ^2_ε term in the covariances.

A. What is γ_0?
B. What is γ_1?
C. What is ρ_1?
D. What is γ_{12}?
E. What is ρ_{12}?
F. What is γ_{11}?
G. What is ρ_{11}?
H. What is γ_{13}?
I. What is ρ_{13}?

Part A: This time series can be written as $Y_t = e_t - \theta\, e_{t-1} - \Theta\, e_{t-12} + \theta\, \Theta\, e_{t-13}$ (see equation 10.1.2 on page 230).

$\gamma_0 = 1 + \theta^2 + \Theta^2 + \theta^2 \times \Theta^2 = (1 + \theta^2) \times (1 + \Theta^2)$.

Part B: γ_1 = covariance $(e_t - \theta\, e_{t-1} - \Theta\, e_{t-12} + \theta\, \Theta\, e_{t-13}, e_{t-1} - \theta\, e_{t-2} - \Theta\, e_{t-13} + \theta\, \Theta\, e_{t-14}) =$

$$- \theta - \theta\, \Theta^2 = - \theta \times (1 + \Theta^2).$$

Part C: $\rho_1 = \gamma_1 / \gamma_0 = - \theta \times (1 + \Theta^2) / (1 + \theta^2) \times (1 + \Theta^2) = - \theta / (1 + \theta^2)$.

Part D: γ_{12} = covariance $(e_t - \theta\, e_{t-1} - \Theta\, e_{t-12} + \theta\, \Theta\, e_{t-13}, e_{t-12} - \theta\, e_{t-13} - \Theta\, e_{t-24} + \theta\, \Theta\, e_{t-25}) =$

$$- \Theta - \theta^2\, \Theta = - \Theta \times (1 + \theta^2).$$

Part E: $\rho_{12} = \gamma_{12} / \gamma_0 = - \Theta \times (1 + \theta^2) / (1 + \theta^2) \times (1 + \Theta^2) = - \Theta / (1 + \Theta^2)$.

Part F: γ_{11} = covariance $(e_t - \theta\, e_{t-1} - \Theta\, e_{t-12} + \theta\, \Theta\, e_{t-13}, e_{t-11} - \theta\, e_{t-12} - \Theta\, e_{t-23} + \theta\, \Theta\, e_{t-24}) = + \Theta \times \theta$.

Part G: $\rho_{11} = \gamma_{11} / \gamma_0 = \Theta \times \theta / (1 + \theta^2) \times (1 + \Theta^2)$.

Part H: γ_{13} = covariance $(e_t - \theta\, e_{t-1} - \Theta\, e_{t-12} + \theta\, \Theta\, e_{t-13}, e_{t-13} - \theta\, e_{t-14} - \Theta\, e_{t-25} + \theta\, \Theta\, e_{t-26}) = + \Theta \times \theta$.

Part I: $\rho_{13} = \gamma_{13} / \gamma_0 = \Theta \times \theta / (1 + \theta^2) \times (1 + \Theta^2)$.

Jacob: Do exam problems give the characteristic polynomial for seasonal models?

Rachel: Yes. A characteristic polynomial of $(1 - \theta x)(1 - \Theta x^{12})$ means $Y_t = e_t - \theta\, e_{t-1} - \Theta\, e_{t-12} + \theta\, \Theta\, e_{t-13}$

** Exercise 20.3: Seasonal moving average process

A seasonal moving average process has a characteristic polynomial of $(1 - \theta x)(1 - \Theta x^{12})$, with $\theta = 0.4$, $\Theta = 0.5$, and $\sigma^2_e = 4$.

A. What is γ_0?
B. What is γ_1?
C. What is ρ_1?
D. What is γ_{12}?
E. What is ρ_{12}?
F. What is γ_{11}?
G. What is ρ_{11}?
H. What is γ_{13}?
I. What is ρ_{13}?

Part A: $\gamma_0 = (1 + \theta^2) \times (1 + \Theta^2) \times \sigma^2_\varepsilon = (1 + 0.16) \times (1 + 0.25) \times 4 = 5.800$.

Part B: $\gamma_1 = -\theta \times (1 + \Theta^2) \times \sigma^2_\varepsilon = -0.4 \times 1.25 \times 4 = -2.000$.

Part C: $\rho_1 - \theta / (1 + \theta^2) = -0.4 / 1.16 = -0.345$.

Part D: $\gamma_{12} = -\Theta \times (1 + \theta^2) \times \sigma^2_\varepsilon = -2.320$.

Part E: $\rho_{12} = -\Theta / (1 + \Theta^2) = -0.5 / 1.25 = -0.400$.

Part F: $\gamma_{11} = +\Theta \times \theta \times \sigma^2_\varepsilon = 0.4 \times 0.5 \times 4 = 0.800$.

Part G: $\rho_{11} = \gamma_{11} / \gamma_0 = 0.13793$.

Part H: $\gamma_{13} = +\Theta \times \theta \times \sigma^2_\varepsilon = 0.4 \times 0.5 \times 4 = 0.800$.

Part I: $\rho_{13} = \gamma_{13} / \gamma_0 = 0.13793$.

** Exercise 20.4: Seasonal ARMA process

A seasonal ARMA process $Y_t = \Phi\ Y_{t-12} + e_t - \theta\ e_{t-1}$ has a variance of the error term σ^2_e.

This is a multiplicative seasonal ARMA(p,q) × (P,Q) process, with p = 0, q = 1, P = 1, Q = 0.

Let *k* be an integer (1, 2, 3, …).

A. What is γ_0?
B. What is ρ_{12}?
C. What is ρ_{13}?
D. What is ρ_{11}?
E. What is ρ_{12k}?
F. What is ρ_{12k+1}?
G. What is ρ_{12k-1}?

Part A: The variance of the multiplicative ARMA process is the product of the autoregressive and moving average parts.

- An MA(1) process has $\gamma_0 = \sigma^2_\varepsilon (1 + \theta^2)$.
- An AR(1) process has $\gamma_0 = \sigma^2_\varepsilon / (1 - \phi^2)$.
- A seasonal AR(1) process has $\gamma_0 = \sigma^2_\varepsilon / (1 - \Phi^2)$.
- A multiplicative seasonal ARMA(p,q) × (P,Q) process has $\gamma_0 = \sigma^2_e \times (1 + \theta^2) / (1 - \Phi^2)$.

Part B: The moving average part of this process has a one-period effect. The 12 month autocorrelation stems from the autoregressive process: $\rho_{12} = \Phi$

Part C: The 13 month autocorrelation is the product of the one-month moving average autocorrelation and the 12 month autoregressive correlation: $\rho_{13} = -\Phi\ \theta / (1 + \theta^2)$

Part D: The 11 month autocorrelation is the product of the moving average autocorrelation for a lag of negative one month and the 12 month autoregressive correlation: $\rho_{13} = -\Phi\ \theta / (1 + \theta^2)$

Jacob: How did we get the moving average autocorrelation for a lag of negative one month?

Rachel: A stationary ARMA process is symmetric: $\rho_k = \rho_{-k}$

Part E: The autocorrelation for a lag of 12k months is Φ^k.

Part F: The autocorrelation for a lag of 12k+1 months is $-\Phi^k\ \theta / (1 + \theta^2)$.

Part G: The autocorrelation for a lag of 12k-1 months is $-\Phi^k\ \theta / (1 + \theta^2)$.

Jacob: For the multiplicative seasonal moving average process, we wrote the process as a combination of error terms to compute the covariance at each lag. Can we do the same with the multiplicative seasonal ARMA process?

Rachel: $Y_t = e_t + \Phi\ Y_{t-12}$ can be written as $Y_t = e_t + \Phi\ e_{t-12} + \Phi^2\ e_{t-24} + \Phi^3\ e_{t-36} + \dots$ Combine this infinite series with the moving average piece to get a single expression of error terms. Compute the covariances to get the formulas in the textbook. Each covariance is a infinite series of Φ terms, which is rewritten as $\Phi^k / (1 - \Phi^2)$.

** Exercise 20.5: Seasonal ARMA process

A seasonal ARMA process $Y_t = \Phi\ Y_{t-12} + e_t - \theta\ e_{t-1}$ has $\Phi = 0.4$, $\theta = 0.5$, and $\sigma^2_e = 4$

A. What is γ_0?
B. What is ρ_{12}?
C. What is ρ_{13}?
D. What is ρ_{11}?
E. What is ρ_{24}?
F. What is ρ_{23}??

Part A: $\gamma_0 = \sigma^2_e \times (1 + \theta^2) / (1 - \Phi^2) = 4 \times (1 + 0.5^2) / (1 - 0.4^2) = 6$

Part B: $\rho_{12} = \Phi = 0.4$

Part C: $\rho_{13} = -\Phi\ \theta / (1 + \theta^2) = -(0.4) \times 0.5 / (1 + 0.5^2) = -0.160$.

Part D: $\rho_{11} = -\Phi\ \theta / (1 + \theta^2) = -(0.4) \times 0.5 / (1 + 0.5^2) = -0.160$.

Part E: $\rho_{24} = \Phi^2 = (0.4)^2 = 0.160$.

Part F: $\rho_{23} = -\Phi^2\ \theta / (1 + \theta^2) = -(0.4)^2 \times 0.5 / (1 + 0.5^2) = -0.064$.

** Exercise 20.6: Seasonal non-stationary $AR(1)_{12}$ process

A store's toy sales in constant dollars follow a seasonal non-stationary $AR(1)_{12}$ process: $Y_t = \Phi Y_{t-12} + e_t$

- Sales are $10,000 in January 20X1 and $100,000 in December 20X1.
- Projected sales are $110,000 in December 20X2.

A. What is Φ?
B. What are projected sales for January 20X2?
C. What are projected sales for January 20X3?

Part A: Using December 20X1 and December 20X2, we have $110,000 = Φ × $100,000 $E(e_t) \Rightarrow \Phi = 1.1$, since the expected residual is zero.

Part B: The forecast for January 20X2 is 1.1 × $10,000 = $11,000.

Part C: The forecast for January 20X3 is 1.1 × $11,000 = $12,100.

(See Cryer and Chan, page 241, last line; $\Phi = 1.1$)

Jacob: How do we make this time series into a stationary process?

Rachel: Take logarithms and first differences to make this process stationary.