TS Module 9 Non-stationary ARIMA time series

(The attached PDF file has better formatting.)

Time series ARIMA processes practice problems

** Exercise 9.1: ARIMA Process

A time series is $Y_t = 1.4Y_{t-1} + 0.1Y_{t-2} – 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

A. Write the time series in terms of W_t (∇Y_t).
B. What is the ARIMA process followed by this time series?
C. What are the coefficients of this ARIMA process?

Part A: Rewrite the ARIMA process as

$Y_t – Y_{t-1} = 0.4Y_{t-1} + 0.1Y_{t-2} – 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

$= 0.4Y_{t-1} – 0.4Y_{t-2} + 0.4Y_{t-2} + 0.1Y_{t-2} – 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

$= 0.4Y_{t-1} – 0.4Y_{t-2} + 0.5Y_{t-2} – 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

$\Rightarrow W_t = 0.4 W_{t-1} + 0.5W_{t-2} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

Jacob: What is the procedure for this transformation?

Rachel: The sum of the coefficients for the Y terms are equal on both sides of the equation.

The left side has a coefficient of 1. The right side has coefficients of 1.4 + 0.1 – 0.5 = 1.

Part B: The time series is an ARIMA(2,1,2) process.

- d = 1: we took one difference (∇Y_t).
- p = 2: we use W_{t-1} and W_{t-2}.
- q = 2: we use e_{t-1} and e_{t-2}.

Part C: The ARIMA coefficients are

$\phi_1 = 0.4$
$\phi_2 = 0.5$
$\theta_1 = –0.3$
$\theta_2 = – 0.2$

** Exercise 9.2: ARIMA Process

A time series is $Y_t = \theta_0 + \alpha_1 \times Y_{t-1} + \alpha_2 \times Y_{t-2} + \alpha_3 \times Y_{t-3} + \beta_1 \times e_t + \beta_2 \times e_{t-1} + \beta_3 \times e_{t-2}$

A. Write the time series in terms of W_t (∇Y_t).
B. What is the ARIMA process followed by this time series?

Part A: Rewrite the ARIMA process as

$Y_t - Y_{t-1} = \theta_0 + (\alpha_1 - 1) \times Y_{t-1} + \alpha_2 \times Y_{t-2} + \alpha_3 \times Y_{t-3} + \beta_1 \times e_t + \beta_2 \times e_{t-1} + \beta_3 \times e_{t-2}$

$Y_t - Y_{t-1} = \theta_0 + [(\alpha_1 - 1) \times Y_{t-1} - (\alpha_1 - 1) \times Y_{t-2}] + [(\alpha_1 - 1) \times Y_{t-2} + \alpha_2 \times Y_{t-2} + \alpha_3 \times Y_{t-3}] + \beta_1 \times e_t + \beta_2 \times e_{t-1} + \beta_3 \times e_{t-2}$

The coefficient of $Y_{t-1} - Y_{t-2}$ is $(\alpha_1 - 1)$.

For this to be an ARIMA(2,1,2) process, we must have

$$(\alpha_1 - 1) + \alpha_2 = -\alpha_3$$
$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

Jacob: What about the β coefficients?

Rachel: Any β coefficients are fine.

- If $\beta_1 = 1$, then $\phi_1 = -\beta_2$ and $\phi_2 = -\beta_3$.
- If $\beta_1 \neq 1$, then $\phi_1 = -\beta_2 / \beta_1$ and $\phi_2 = -\beta_3 / \beta_1$.

** Exercise 9.3: Combining error terms

Suppose $Y_t = M_t + e_t$ and $M_t = M_{t-1} + \epsilon_t$

A. Write Y_t as a function of M_{t-1} and error terms.
B. What type of time series is M_t?
C. What type of time series is Y_t?
D. What is ∇Y_t (the first difference of Y_t)?
E. What is the variance of Y_t?
F. What is the variance of ∇Y_t?
G. What is the covariance of ∇Y_t and ∇Y_{t-1}?
H. What is ρ_1, the autocorrelation of ∇Y_t and ∇Y_{t-1}?

Part A: $Y_t = M_t + e_t = M_{t-1} + e_t + \epsilon_t$

Part B: M_t is a random walk.

Part C: $Y_t = M_{t-1} + e_t + \epsilon_t = Y_{t-1} + \epsilon_t + e_t - e_{t-1}$. This is a random walk with a more complex error term.

Part D: $\nabla Y_t = Y_t - Y_{t-1} = M_{t-1} + e_t + \epsilon_t - (M_{t-1} + e_{t-1}) = \epsilon_t + e_t - e_{t-1}$

Part E: If the random walk has no beginning, the variance is infinite, so it does not exist. If the random walk has a beginning, the variance depends on the period.

Part F: The variance of ∇Y_t = var($\epsilon_t + e_t - e_{t-1}$). The three random variables are independent, so the variance = $2\sigma^2_e + \sigma^2_\varepsilon$.

Part G: The covariance of ∇Y_t and ∇Y_{t-1} is covariance ($\epsilon_t + e_t - e_{t-1}$, $\epsilon_{t-1} + e_{t-1} - e_{t-2}$) = $-\sigma^2_e$.

Part H: The autocorrelation of ∇Y_t and ∇Y_{t-1} (ρ_1) is $-\sigma^2_e / 2\sigma^2_e + \sigma^2_\varepsilon = -1 / (2 + \sigma^2_\varepsilon / \sigma^2_e)$. This is equation 5.1.10 on page 90.

** Exercise 9.4: IMA(1,1) process

Each of the following time series is an IMA(1,1) process. What is the value of θ for each time series?

A. $Y_t = Y_{t-1} + e_t - 0.4e_{t-1}$
B. $Y_t = Y_{t-1} - e_t - 0.4e_{t-1}$
C. $Y_t = Y_{t-1} + 0.4e_t - 0.4e_{t-1}$
D. $Y_t = Y_{t-1} - 0.4e_t - 0.4e_{t-1}$

Part A: The first difference of an IMA(1,1) is an MA(1) process.

The first difference of this time series is $e_t - 0.4e_{t-1}$, which is an MA(1) process with $\theta = 0.4$.

Part B: The first difference of this time series is $- e_t - 0.4e_{t-1}$. Use a change of the error term $\epsilon_{t'} = -\epsilon_t$, which gives a first difference of $+ e_{t'} + 0.4e_{t'-1}$, which is an MA(1) process with $\theta = -0.4$.

Part C: The first difference of this time series is $0.4\ e_t - 0.4e_{t-1}$. Use a change of the error term $\epsilon_{t'} = 2.5\epsilon_t$, which gives a first difference of $+ e_{t'} - e_{t'-1}$, which is an MA(1) process with $\theta = 1$.

Part D: The first difference of this time series is $-0.4\ e_t - 0.4e_{t-1}$. Use a change of the error term $\epsilon_{t'} = -2.5\epsilon_t$, which gives a first difference of $+ e_{t'} + e_{t'-1}$, which is an MA(1) process with $\theta = -1$.

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** Exercise 9.5: ARI(1,1) process

The time series $Y_t = \theta_0 + \alpha\ Y_{t-1} + \beta\ Y_{t-2} + e_t$ is an ARI(1,1) process.

A. Write the time series in terms of W_t (∇Y_t).
B. What is the relation of α and β?
C. What is the value of ϕ for this ARI(1,1) process?

Part A: $W_t = \nabla Y_t = Y_t - Y_{t-1} = \theta_0 + (\alpha - 1) \times Y_{t-1} + \beta\ Y_{t-2} + e_t$

Part B: If $\alpha - 1 = -\beta$, we can write the time series as $Y_t - Y_{t-1} = \theta_0 + (-\beta) \times (Y_{t-1} - Y_{t-2}) + e_t$

Part C: $\phi = -\beta$

A. $Y_t = Y_{t-1} - (1 + \phi)\ Y_{t-2} + e_t$
B. $Y_t = Y_{t-1} + \phi\ Y_{t-2} + e_t$
C. $Y_t = Y_{t-1} - \phi\ Y_{t-2} + e_t$
D. $Y_t = (1 + \phi)\ Y_{t-1} - \phi\ Y_{t-2} + e_t$
E. $Y_t = (1 + \phi)\ Y_{t-1} + \phi\ Y_{t-2} + e_t$

** Exercise 9.6: Time series process

A time series is $Y_t = \theta_0 + 1.75\ Y_{t-1} - 0.75Y_{t-2} + e_t$ is an ARI(2,1) process.

A. Write the time series in terms of W_t (∇Y_t).
B. What is the value of ϕ_1 for this ARI(2,1) process?
C. What is the value of ϕ_2 for this ARI(2,1) process?

Part A: $W_t = \nabla Y_t = Y_t - Y_{t-1} = \theta_0 + (\alpha - 1) \times Y_{t-1} + \beta\ Y_{t-2} + e_t$

Part B: If $\alpha - 1 = -\beta$, we can write the time series as $Y_t - Y_{t-1} = \theta_0 + (-\beta) \times (Y_{t-1} - Y_{t-2}) + e_t$

Part C: $\phi = -\ \beta$