Corporate Finance Mod 21: Options, binomial tree pricing method, practice problems

** Exercise 21.1: Binomial tree pricing method

- A stock is trading at $100.
- Over the next six months, the stock price will increase 20% or decrease 20%.
- Six month European put and call options trade with exercise prices of $110.
- The risk-free interest rate is 8% per annum compounded semi-annually, or 4% each six months.

A. What is the risk-neutral probability that the stock price will increase over the next six months?
B. What is the value of the call option if the stock price increases? If the stock price decreases?
C. What is the delta of the call option?
D. What is the replicating portfolio for the call option?
E. What is the price of the call option?
F. In a risk-neutral world, what is the expected value of the call option at the expiration date?
G. What is the present value of the call option at inception?
H. What is the value of the put option if the stock price increases? If the stock price decreases?
I. What is the delta of the put option?
J. What is the replicating portfolio for the put option?
K. What is the price of the put option?
L. In a risk-neutral world, what is the expected value of the put option at the expiration date?
M. What is the present value of the put option at inception?
N. Verify that the put and call values satisfy the put call parity relation.

Part A: The risk-neutral probability of an upward movement in the stock price is (r_f – d) / (u – d), where r_f is the risk-free interest rate, *u* is the upward movement in the stock price, and *d* is the downward movement.

$$p = \frac{r_f - d}{u - d}$$

= (0.04 – (–0.20)) / (0.20 – (–0.20)) = 60%.

Part B: If the stock price increases to $100 × 1.20 = $120, the call option is worth $120 – $110 = $10.

If the stock price decreases to $100 × 0.80 = $80, the call option is worth zero.

Part C: The delta of the call option is the difference in price of the call option divided by the difference in price of the stock: ($10 – $0) / ($120 – $80) = 25%.

Part D: The call option delta is 25%, so the replicating portfolio has 25% × S. If the stock price increases to $120, 25% × S is $30. The call option is worth $10, so the replicating portfolio at the expiration date is 25% × S – $20, and the replicating portfolio now is the present value: 25% × S – $20 / 1.04 = 25% × S – $19.23. *Verify:* if the stock price decreases to $80, 25% × S – $20 is $20 – $20 = $0.

Part E: The price of the call option is 25% × $100 – $20 / 1.04 = $5.77.

Part F: In a risk-neutral world, the expected value of the call option at the expiration date is

60% × $10 + (1 – 60%) × $0 = $6.

Part G: The present value of the call option at inception is $6 / 1.04 = $5.77.

Part H: If the stock price increases to $100 × 1.20 = $120, the put option is worth zero.

If the stock price decreases to \$100 × 0.80 = \$80, the put option is worth \$110 – \$80 = \$30.

Part I: The delta of the put option is the difference in price of the put option divided by the difference in price of the stock: (\$0 – \$30) / (\$120 – \$80) = -75%.

Part J: The put option delta is –75%, so the replicating portfolio has –75% × S. If the stock price increases to \$120, –75% × S is –\$90. The call option is worth \$0, so the replicating portfolio at the expiration date is –75% × S + \$90, and the replicating portfolio now is the present value: –75% × S + \$90 / 1.04 = –75% × S + \$86.54. *Verify:* if the stock price decreases to \$80, –75% × S + \$90 is –\$60 + \$90 = \$30.

Part K: The price of the put option is –75% × \$100 + \$90 / 1.04 = \$11.54.

Part L: In a risk-neutral world, the expected value of the put option at the expiration date is

60% × \$0 + (1 – 60%) × \$30 = \$12.

Part M: The present value of the put option at inception is \$12 / 1.04 = \$11.54.

Part N: The put call parity relation is c + PV(X) = p + S

\$5.77 + \$110 / 1.0 4 = \$111.54 = \$11.54 + \$100

** Exercise 21.2: Up and Down Movements

Jacob and Rachel own six month European put and call options with strike prices of $80 on different stocks.

- Both stocks now trade at $80.
- The risk-free rate is 1% each month (or 1.01^{12} for the year).

Jacob's stock moves up or down 10% each month, and Rachel's stock moves up or down 20% each month.

A. Whose stock has higher volatility?
B. Whose call option is worth more?
C. Whose put option is worth more?

Part A: The volatility of Rachel's stock is higher than that of Jacob' stock. The future value of Rachel's stock has a more diffuse probability distribution: specifically, a lognormal distribution with a higher σ.

Part B: Rachel's call option is worth more. Higher volatility raises the value of the call option.

Part C: Rachel's put option is worth more. Higher volatility raises the value of the put option.

We examine this relation by intuition, binomial tree pricing method (binomial lattice), Black-Scholes formula.

Intuition: Rachel's stock price may increase a lot or decrease a lot. If it increases a lot, Rachel has a large gain from her call option; if it decreases a lot, she does not lose anything on her call option. From her put option, Rachel gains a lot if the stock price decreases a lot but doesn't lose if the stock price increases a lot.

Jacob's stock price may increase less or decrease less. If it increases, Jacob has a smaller gain from the call option; if it decreases, he does not lose anything on the call option. From the put option, Jacob gains less if the stock price decreases but doesn't lose if the stock price increases.

Question: Does the effect of volatility on the option price depend on the stock price and the strike price?

Answer: The principle is true regardless of the stock price and strike price: higher volatility means a higher option price. The exact effect depends on the stock price and strike price. The effect is greatest when the stock price equals the present value of the strike price: the option is at the money. The effect is least when the option is far in-the-money or far out-of-the-money.

Binomial Lattice: Working out a six period lattice with pencil and paper is cumbersome. To simplify, we examine a one month binomial lattice.

The present value of the European call option is

the value of the call option at the expiration date if the stock price moves up
× the risk-neutral probability of moving up / the risk-free rate

- For Jacob, if the stock price moves up 10%, the call option is worth $88 – $80 = $8. The risk-neutral probability of moving up is (1.01 – 0.90) / (1.10 – 0.90) = 55.00%. The present value of the call option is $8 × 55% / 1.01 = $4.36.
- For Rachel, if the stock price moves up 20%, the call option is worth $96 – $80 = $16. The risk-neutral probability of moving up is (1.01 – 0.80) / (1.20 – 0.80) = 52.50%. The present value of the call option is $16 × 52.5% / 1.01 = $8.32.

Rachel's call option is worth almost twice as much. If the stock price moves up, her call option pays twice as much, but the risk-neutral probability that her stock moves up is slightly lower.

Black-Scholes formula: A higher stock price volatility σ increases d_1 and decreases d_2.

- For a call option, this increases $S \times N(d_1)$ and decreases $PV(X) \times N(d_2)$, so it increases the present value of the call option.
- For a put option, this decreases $S \times N(-d_1)$ and increases $PV(X) \times N(-d_2)$, so it increases the present value of the put option.

Question: The volatility is the total standard deviation of the stock price. The expected return of a stock depends on its systematic risk. If Rachel's stock has higher volatility and Jacob's stock has higher systematic risk, whose options are worth more?

Answer: The expected return of a stock depends on its systematic risk, not its total volatility. The value of an option depends on its total volatility, not its systematic risk. The principle:

For expected returns, look at systematic risk; for option prices, look at volatility (total risk).

** Exercise 21.3: Real vs risk-neutral probabilities

A stock has a CAPM beta of one. The risk-free rate is 4% and the market risk premium is 8%. Over the next year, the stock price will either increase 20% or decrease 20%.

A. What is the expected return on the stock?
B. What is the probability that the stock price will increase during the year?
C. What is the risk-neutral probability that the stock price will increase during the year?

Part A: The expected return on the stock is 4% + 1.00 × 8% = 12%.

Part B: Let the p be the probability that the stock price will increase.

$12\% = p \times 20\% + (1 - p) \times -20\% \Rightarrow p = (12\% - (-20\%)) / (20\% - (-20\%)) = 80\%$

Part B: Let the p be the risk-neutral probability that the stock price will increase.

$4\% = p \times 20\% + (1 - p) \times -20\% \Rightarrow p = (4\% - (-20\%)) / (20\% - (-20\%)) = 60\%$