TS Module 6 Stationary autoregressive processes

(The attached PDF file has better formatting.)

Time series autoregressive processes practice problems

** Exercise 6.1: Stationarity of AR(2) process

A. What are the three conditions for an AR(2) process to be stationary?
B. Given ϕ_1, what values of ϕ_2 create a stationary AR(2) process?

Part A: The three conditions are

1. $\phi_1 + \phi_2 < 1$
2. $\phi_2 - \phi_1 < 1$
3. $|\phi_2| < 1$

(See Cryer and Chan page 72, equation 4.3.11)

Part B: If ϕ_1 is given, the conditions are

1. $\phi_2 < 1 - \phi_1$
2. $\phi_2 < 1 + \phi_1$
3. $|\phi_2| < 1$

Illustration: If $\phi_1 = 0.9$, then ϕ_2 must be less than 0.1 and more than –1.

*Question 6.2: Characteristic polynomial

An AR(2) process is $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$.

What is the characteristic polynomial for this time series?

A. $\phi(x) = 1 + \phi_1(x) + \phi_2(x^2)$
B. $\phi(x) = 1 + \phi_2(x) + \phi_1(x^2)$
C. $\phi(x) = 1 - \phi_1(x) - \phi_2(x^2)$
D. $\phi(x) = 1 - \phi_2(x) + \phi_1(x^2)$
E. $\phi(x) = 1 + \phi_2(x) - \phi_1(x^2)$

Answer 6.2: C

(See Cryer and Chan page 71, equation 4.3.9)

The characteristic polynomial reverses the sign of the autoregressive coefficient.

** Exercise 6.3: AR(2) autocorrelations

An AR(2) process with mean zero is $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t$

A. What is γ_0, the variance of Y_t?
B. What is γ_1, the covariance of Y_t and Y_{t-1}?
C. What is γ_2, the covariance of Y_t and Y_{t-2}?
D. What is ρ_1, the autocorrelation of lag 1?
E. What is ρ_2, the autocorrelation of lag 2?

Part A: Multiply the expression for the AR(2) process by Y_{t-1} and take expectations to get

$$E(Y_t \times Y_{t-1}) = \phi_1 E(Y_{t-1} \times Y_{t-1}) + \phi_2 E(Y_{t-2} \times Y_{t-1}) + E(\epsilon_t \times Y_{t-1})$$

The time series observations are uncorrelated with the error terms in subsequent periods, so $E(\epsilon_t, Y_{t-1}) = 0$.

$E(Y_t, Y_{t-1}) = \gamma_1$, $E(Y_{t-1}, Y_{t-1}) = \gamma_0$, and $E(Y_{t-2}, Y_{t-1}) = \gamma_1$, so

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$$

Divide this equation by γ_0 to get

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1$$

ρ_o is the correlation of a random variable with itself, which is always 1, so we derive that

$$\rho_o = \theta_1 + \theta_2 \rho_1 \Rightarrow \rho_1 = \theta_1 / (1 - \theta_2)$$

Part B: Multiply the expression for the AR(2) process by Y_{t-2} and take expectations to get

$$E(Y_t \times Y_{t-2}) = \phi_1 E(Y_{t-1} \times Y_{t-2}) + \phi_2 E(Y_{t-2} \times Y_{t-2}) + E(\epsilon_t \times Y_{t-2}) \Rightarrow$$

The time series observations are uncorrelated with the error terms in subsequent periods, so $E(\epsilon_t, Y_{t-2}) = 0$.

$E(Y_t, Y_{t-2}) = \gamma_2$, $E(Y_{t-1}, Y_{t-2}) = \gamma_1$, and $E(Y_{t-2}, Y_{t-2}) = \gamma_0$, so

$$\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0$$

Divide by γ_0 to get

$$\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0 = \phi_2 + \phi_1^2 / (1 - \phi_2)$$

Cryer and Chan write this as

$$\rho_2 = [\ \phi_2 (1 - \phi_2) + \phi_1^2\ ] / (1 - \phi_2)$$