Module 7: Mixed autoregressive moving average (ARMA) practice problems

(The attached PDF file has better formatting.)

** Exercise 7.1: Mixed autoregressive moving average (ARMA) process

An ARMA(1,1) process has $\sigma^2 = 1$, $\theta = -0.4$, and $\phi = 0.8$.

A. What is the value of γ_0?
B. What is the value of γ_1?
C. What is the value of ρ_1?
D. What is the value of ρ_2?

ARMA(1,1) means the process has 1 ϕ parameter and 1 θ parameter. Know equations 4.4.3, 4.4.4, and 4.4.5 on page 78. These seem complex at first. Review the derivation in the textbook, so you recall the formulas.

After period 1, γ_k and ρ_k have exponential decay. ρ_1 is γ_1 / γ_0. γ_1 is a simple function of γ_0, ϕ_1, and θ. The final exam problems test ρ_1 most frequently.

Part A: For an ARMA(1,1) process, $\gamma_0 = \dfrac{(1 - 2\phi\theta + \theta^2)}{1 - \phi^2}\sigma_\varepsilon^2$

$= (1 - 2 \times 0.8 \times -0.4 + (-0.4)^2) / (1 - (0.8)^2) = 5$

(See Cryer and Chan, chapter 4, page 78, equation 4.4.4)

Part B: $\gamma_1 = \phi\ \gamma_0 - \theta\ \sigma^2_\varepsilon = 0.8 \times 5 - (-0.4) \times 1 = 4.4$

See Cryer and Chan, chapter 4, page 78, equation 4.4.3

Jacob: How do we derive these formulas?

Rachel: γ_0 is the variance of $Y_t = E(Y_t^2)$.

To avoid superfluous parameters, subtract the mean from all values of the time series.

The variance of Y_t, the covariances, and the autocorrelations do not change, and we don't have to include a μ parameter in the derivations.

For an ARMA(1,1) process with a mean of zero, $Y_t = \phi \times Y_{t-1} + \epsilon_t - \theta \times \epsilon_{t-1} \Rightarrow$

$\gamma_0 = E(Y_t^2) = E(Y_t \times (\phi \times Y_{t-1} + \epsilon_t - \theta \times \epsilon_{t-1}) = \phi \times E(Y_t \times Y_{t-1}) + E(\epsilon_t \times Y_t) - \theta \times E(\epsilon_{t-1} \times Y_t) \Rightarrow$

$$\gamma_0 = (\phi \times \gamma_1) + E(\epsilon_t \times Y_t) - \theta \times E(\epsilon_{t-1} \times Y_t)$$

Also: $\gamma_1 = E(Y_{t-1} \times Y_t) = E(Y_{t-1} \times (\phi \times Y_{t-1} + \epsilon_t - \theta \times \epsilon_{t-1}) = \phi \times E(Y_{t-1} \times Y_{t-1}) + E(\epsilon_t \times Y_{t-1}) - \theta \times E(\epsilon_{t-1} \times Y_{t-1})$.

For a stationary time series, $E(Y_{t-1} \times Y_{t-1}) = E(Y_t \times Y_t) = \gamma_0$ and $E(\epsilon_{t-1} \times Y_{t-1}) = E(\epsilon_t \times Y_t)$, so

$$\gamma_1 = (\phi \times \gamma_0) + E(\epsilon_t \times Y_{t-1}) - \theta \times E(\epsilon_t \times Y_t)$$

If we can solve for $E(\epsilon_t \times Y_t)$ and $E(\epsilon_{t-1} \times Y_t)$ in terms of the ARMA(1,1) parameters ϕ, θ, and σ^2_t, we have two equations in two unknowns (γ_0 and γ_1).

ϵ_t is uncorrelated with Y_{t-1} and with ϵ_{t-1}, so

$$E(\epsilon_t \times Y_t) = E(\epsilon_t \times (\phi \times Y_{t-1} + \epsilon_t - \theta \times \epsilon_{t-1})) = 0 + \sigma^2_t + 0 = \sigma^2_t$$

Similarly, ϵ_{t-1} is uncorrelated with ϵ_t and the correlation of ϵ_{t-1} with Y_{t-1} is σ^2_t, so

$$E(\epsilon_{t-1} \times Y_t) = E(\epsilon_{t-1} \times (\phi \times Y_{t-1} + \epsilon_t - \theta \times \epsilon_{t-1})) = \phi \times \sigma^2_t + 0 - \theta \times \sigma^2_t = (\phi - \theta) \times \sigma^2_t$$

We now have two equations in two unknowns:

$$\gamma_0 = (\phi \times \gamma_1) + \sigma^2_t - (\theta \times (\phi - \theta) \times \sigma^2_t) \Rrightarrow$$
$$\gamma_0 = (\phi \times \gamma_1) + ([1 - \theta \times (\phi - \theta)] \times \sigma^2_t)$$

$$\gamma_1 = (\phi \times \gamma_0) - (\theta \times \sigma^2_t)$$

We use the expression for γ_1 in the formula for γ_0 to get

$$\gamma_0 = (\phi \times [(\phi \times \gamma_0) - (\theta \times \sigma^2_t)]) + ([1 - \theta \times (\phi - \theta)] \times \sigma^2_t) \Rrightarrow$$

$$(1 - \phi^2) \times \gamma_0 = (\phi \times (\theta \times \sigma^2_t)]) + ([1 - \theta \times (\phi - \theta)] \times \sigma^2_t) \Rrightarrow$$

$$\gamma_0 = (1 - 2\theta\phi - \theta^2) \times \sigma^2_t) / (1 - \phi^2)$$

We express γ_1 as a function of ϕ, θ, and σ^2_t:

$$\gamma_1 = (\phi \times \gamma_0) - (\theta \times \sigma^2_t) = \{ [\phi \times (1 - 2\theta\phi - \theta^2)] - [\theta \times (1 - \phi^2)] \} \times \sigma^2_t) / (1 - \phi^2) \Rrightarrow$$

$$\gamma_1 = [(\phi - \theta - 2 \theta \phi^2 + \theta \phi^2) \times] \sigma^2_t / (1 - \phi^2) \Rrightarrow$$

$$\gamma_1 = [(\phi - \theta) \times (1 - 2\theta\phi^2 + \theta\phi^2)] \times \sigma^2_t / (1 - \phi^2)$$

We express ρ_1 as a function of ϕ and θ:

$\rho_1 = \gamma_1 / \gamma_0$. Both γ_0 and γ_1 have the multiplicative term $\sigma^2_t / (1 - \phi^2)$, which cancels out of the ratio for ρ_1:

$$\rho_1 = [(\phi - \theta) \times (1 - 2\theta\phi^2 + \theta\phi^2)] / (1 - 2\theta\phi - \theta^2)$$

The moving average effect dies out after one period, and the autoregressive effect decays exponentially, so

$$\rho_k = \rho_{k-1} \times \phi \Rrightarrow$$

$$\rho_k = \rho_1 \times \phi^{k-1}$$

See Cryer and Chan, chapter 4, pages 77-78, equations 4.4.2-4.4.4. Cryer and Chan write the formulas for γ_0, γ_1, and ρ_k. Rachel shows the derivation in more detail. Once you have read Rachel's dialogue, you can see the intuition for the expressions. The equations appears on final exam problems, so follow Rachel's derivation.

Part C: For an ARMA(1,1) process: $\rho_k = \dfrac{(1-\phi\theta)(\phi-\theta)}{1-2\phi\theta-\theta^2}\phi^{k-1}$

Since we have solved for γ_0 and γ_1, we use the relation $\rho_1 = \gamma_1 / \gamma_0 = 4.4 / 5 = 0.88$

In practice, you use the autocorrelations ρ_k more frequently than the autocovariances γ_k.

Part D: The autocorrelations ρ_k decline geometrically at a rate $\phi \Rightarrow \rho_2 = \rho_1 \times \phi = 0.88 \times 0.8 = 0.704$

In the module on method of moments, we start with sample autocorrelations of 0.88 and 0.704 and derive estimates of the ϕ and θ parameters.

** Exercise 7.2: Moving average and autoregressive processes

An autoregressive process with mean zero is $Y_t = \phi\ Y_{t-1} + \epsilon_t$.

Re-write this time series as a moving average process of infinite lag.

Solution 7.2:

$Y_t = e_t + \phi\ Y_{t-1}$
$= e_t + \phi\ (e_{t-1} + \phi\ Y_{t-2})$
$= e_t + \phi\ e_{t-1} + \phi^2\ (e_{t-2} + \phi\ Y_{t-3})$

Continue expanding to eliminate all the Y_t and remain with an infinite series of ϵ's.

$Y_t = e_t + \phi\ e_{t-1} + \phi^2\ e_{t-2} + \phi^3 \epsilon_3 + \ldots$

See Cryer and Chan, chapter 4, page 70, equation 4.3.8

Cryer and Chan refer to the coefficients of the ϵ's as ψ weights or filter representation.

Jacob: Why would we want to express an AR(1) process as a moving average process of infinite lag?

Rachel: The values of an AR(1) process are autocorrelated. The ϵ terms are independent, so we can easily compute the variance of the autoregressive process as

$$\text{variance } (Y_t) = (1 + \phi + \phi^2 + \phi^3 + \ldots\ ) \times \sigma^2{}_t = \sigma^2{}_t\ /\ (1 - \phi^2).$$

* Question 7.3: ARMA(1,1)

An ARMA(1,1) process has $\rho_1 = 0$. Which of the following is true?

A. $\phi_1 = \theta_1$ or $\phi_1 \times \theta_1 = 1$
B. $\phi_1 = \theta_1$ and $\phi_1 \times \theta_1 = 1$
C. $\phi_1 = -\theta_1$ or $\phi_1 \times \theta_1 = -1$
D. $\phi_1 = -\theta_1$ and $\phi_1 \times \theta_1 = -1$
E. $\phi_1 = \theta_1$ and $\phi_1 \times \theta_1 = -1$

Answer 7.3: A

For an ARMA(1,1) process:

$$\rho_k = \frac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\phi^{k-1} \qquad \text{for } k \geq 1$$

Notation: For AR(1), MA(1), and ARMA(1,1) processes, Cryer and Chan drop the subscripts on the parameters, using ϕ and θ instead of ϕ_1 and θ_1. The final exam problems may use either notation.

Intuition: If the residual in period t increases one unit:

- Moving average: θ_1 causes the forecast to decrease θ_1 units
- Autoregressive: ϕ_1 causes the forecast to increase ϕ_1 units

If $\phi_1 = \theta_1$, these two effects offset each other. A change in the period t value does not affect the period t+1 value, so $\rho_1 = 0$.

θ_1 and $1/\theta_1$ produce the same autocorrelation function.

$\phi_1 = \theta_1$ has the same effect as $\phi_1 = 1/\theta_1$ or $\phi_1 \times \theta_1 = 1$.