TS Module 9 Non-stationary ARIMA time series

(The attached PDF file has better formatting.)

Time series ARIMA processes practice problems

** Exercise 9.1: ARIMA(0,1,1) process

A. What is an ARIMA(0,1,1) process?
B. Write an ARIMA(0,1,1) process as a series of error terms.
C. What is the variance of an ARIMA(0,1,1) process?
D. What is the mean of an ARIMA(0,1,1) process?

Part A: An ARIMA(0,1,1) process is a once-integrated MA(1), called an IMA(1,1) process in the text.

The first difference of an ARIMA(0,1,1) process is an MA(1) process.

If Y_t is an ARIMA(0,1,1) process = an IMA(1,1) process, then

$$W_t = \Delta Y_t = Y_t - Y_{t-1} = e_t - \theta e_{t-1} \quad \Rrightarrow$$
$$Y_t = Y_{t-1} + e_t - \theta e_{t-1}$$

Part B: Express Y_t in terms of Y_{t-1} and Y_{t-1} in terms of Y_{t-2}:

- $Y_t = Y_{t-1} + \epsilon_t - \theta\ \epsilon_{t-1}$.
- $Y_{t-1} = Y_{t-2} + \epsilon_{t-1} - \theta\ \epsilon_{t-2}$.
- $\Rrightarrow Y_t = Y_{t-2} + \epsilon_t + \epsilon_{t-1} - \theta\ \epsilon_{1-1} - \theta\ \epsilon_{1-2} = Y_{t-2} + \epsilon_t + (1 - \theta)\ \epsilon_{1-1} - \theta\ \epsilon_{1-2}$.

We continue in this fashion to get

$$Y_t = \epsilon_t + (1 - \theta)\ \epsilon_{1-1} + (1 - \theta)\ \epsilon_{1-2} + (1 - \theta)\ \epsilon_{1-3} + \dots$$

Part C: The error terms ϵ_j are independent with a variance of σ^2_t for each one. The variance of Y_t is

$$\sigma^2_t + (1 - \theta)^2 \times \sigma^2_t + (1 - \theta)^2 \times \sigma^2_t + (1 - \theta)^2 \times \sigma^2_t + \dots = \text{infinity}.$$

Part D: The mean of each error term ϵ_j is zero, so one is tempted to say that the mean of Y_t is zero. But the sum of an infinite number of random variables, each of which has a mean of zero, is not necessarily zero.

If the observed value of Y_t is 1, the expected value of Y_{t+1} is 1; if the observed value of Y_t is 2, the expected value of Y_{t+1} is 2. Given any observed value in the current period, the expected values in future periods differ. The values do not regress toward any point; that is, they have no mean.

Jacob: What is the drift of an ARIMA process?

Rachel: The drift of an ARIMA process with d = 1 is the mean of the underlying ARMA process. If the MA(1) process has a mean of μ, the expected value of the ARIMA(0,1,1) process increases by μ each period.

ARIMA processes with fixed starting points.

Cryer and Chan begin with non-stationary processes that have a fixed starting point. If the time series has no starting point, its mean and variance are not defined. For an infinite ARIMA process, Y_t has no mean or variance. Each entry in the non-stationary time series is the sum of an infinite number of random variables that do not die out. To explain the pattern of these processes, we assume they begin at some time t.

This is not really a restrictive condition. Most commonly, we forecast future values of a time series based on historical observations. Before the first observation, we assume all values of the time series are zero. We model the evolution of the mean and variance of the process.

**Exercise 9.2: ARIMA(0,1,1) process

Suppose $Y_t = Y_{t-1} + e_t - \theta e_{t-1}$

with $Y_t = 0$ for $t < 1$.

A. What is the variance of Y_1?
B. What is the variance of Y_2?
C. What is the variance of Y_T?

Part A: $Y_1 = Y_0 + \epsilon_1 - \theta\,\epsilon_0 = 0 + \epsilon_1 - \theta\,\epsilon_0$

The variance of ϵ is σ^2_ε, so the variance of Y_1 is $(1 + \theta^2) \times \sigma^2_t$.

Part B: $Y_2 = Y_1 + \epsilon_2 - \theta\,\epsilon_1$

$Y_1 = Y_0 + \epsilon_1 - \theta\,\epsilon_0 = 0 + \epsilon_1 - \theta\,\epsilon_0$, so $Y_2 = \epsilon_2 + (1 - \theta)\,\epsilon_1 - \theta\,\epsilon_0$

The variance of ϵ is σ^2_ε, so the variance of Y_2 is $(1 + (1 - \theta)^2 + \theta^2) \times \sigma^2_t$.

Part C: We continue this process *T* times, giving the variance of $Y_2 = (1 + (T-1) \times (1 - \theta)^2 + \theta^2) \times \sigma^2_t$

Jacob: For an ARMA process, we evaluate the variance of Y_t, where *t* may be any element of the time series. Each element has the same mean, so it is a random variable, which has a variance.

For this ARIMA(0,1,1) process, Y_1, Y_2, and Y_T are specific observations; they are values, not random variables. How can they have variances?

Rachel: We have formulated the exercise as Cryer and Chan do in their textbook. We are standing at time t=0, and we are projecting the values at times t=1, t=2, and t=T. Now these are random variables; each has a variance.

Jacob: If we don't specify that $Y_t = 0$ for $t < 1$, what is the variance of Y_t?

Rachel: An ARIMA(0,1,1) process is not stationary and has no variance.

Cryer and Chan, P94: equation 5.2.7, for IMA(1,1) process, is

$$Var(Y_t) \quad \left[1 + \theta^2 + (1-\theta)^2 (t+m)\right]\sigma_e^2$$

Cryer and Chan assume that $Y_t = 0$ for periods $t < -m$. They assume we are now at Period 0, looking forward to estimate the variances at Periods 1, 2, 3, …. The first observed value is Period 1, but the process has been going on since Period –(m-1). We don't observe the value at Period 0, but we know the time series process.

The Cryer and Chan scenario is complex, making it difficult to solve problems by first principles. Final exam problems use the scenario here: $Y_t = 0$ for $t < 1$.

** Exercise 9.3: ARIMA(0,1,1) process

Suppose $Y_t = Y_{t-1} + e_t - \theta e_{t-1}$

with $\theta = 0.4$ and $\sigma^2_\varepsilon = 4$ for $t > 0$ and $Y_t = 0$ for $t < 1$.

A. What is the variance of Y_1?
B. What is the variance of Y_2?
C. What is the variance of Y_3?

Part A: $Y_1 = Y_0 + \epsilon_1 - \theta\ \epsilon_0 = 0 + \epsilon_1 - \theta\ \epsilon_0$

The variance of ϵ is 4, so the variance of Y_1 is $(1 + 0.4^2) \times 4 = 4.64$

Part B: $Y_2 = Y_1 + \epsilon_2 - \theta\ \epsilon_1$

$Y_1 = Y_0 + \epsilon_1 - \theta\ \epsilon_0 = 0 + \epsilon_1 - \theta\ \epsilon_0$, so $Y_2 = \epsilon_2 + (1 - \theta)\ \epsilon_1 - \theta\ \epsilon_0$

The variance of ϵ is 4, so the variance of Y_2 is $(1 + (1 - 0.4)^2 + 0.4^2) \times 4 = 6.08$

Part C: The variance of Y_T is $(1 + (T - 1) \times (1 - \theta)^2 + \theta^2) \times \sigma^2_t$.

The variance of Y_3 is $(1 + 2 \times (1 - 0.4)^{22} + 0.4^2) \times 4 = 7.5200$

** Exercise 9.4: ARIMA Process

A time series is $Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + (1 - \alpha_1 - \alpha_2) Y_{t-3} + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2}$

A. Write the time series in terms of W_t (∇Y_t).
B. What ARIMA process is this time series?
C. What are the ϕ and θ coefficients of this ARIMA process?

Part A: Rewrite the ARIMA process as

$Y_t - Y_{t-1} = (\alpha_1 - 1) Y_{t-1} + \alpha_2 Y_{t-2} + (1 - \alpha_1 - \alpha_2) Y_{t-3} + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2}$

$Y_t - Y_{t-1} = (\alpha_1 - 1) Y_{t-1} - (\alpha_1 - 1) Y_{t-2} + (\alpha_1 - 1) Y_{t-2} + \alpha_2 Y_{t-2} + (1 - \alpha_1 - \alpha_2) Y_{t-3} + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2}$

$Y_t - Y_{t-1} = (\alpha_1 - 1) Y_{t-1} - (\alpha_1 - 1) Y_{t-2} + (\alpha_1 + \alpha_2 - 1) Y_{t-2} - (\alpha_1 + \alpha_2 - 1) Y_{t-3} + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2}$

$W_t = (\alpha_1 - 1) W_{t-1} + (\alpha_1 + \alpha_2 - 1) W_{t-2} + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2}$

Jacob: What is the procedure for this transformation?

Rachel: The sum of the coefficients for the Y terms are equal on both sides of the equation.

The left side has a coefficient of 1. The right side has coefficients of $\alpha_1 + \alpha_2 + (1 - \alpha_1 - \alpha_2) = 1$.

Part B: The time series is an ARIMA(2,1,2) process.

- d = 1: we took one difference (∇Y_t).
- p = 2: we use W_{t-1} and W_{t-2}.
- q = 2: we use e_{t-1} and e_{t-2}.

Part C: The ARMA coefficients are

$\phi_1 = (\alpha_1 - 1)$
$\phi_2 = (\alpha_1 + \alpha_2 - 1)$
$\theta_1 = -\beta_1$
$\theta_2 = -\beta_2$

Illustration: A time series is $Y_t = 1.4Y_{t-1} + 0.1Y_{t-2} - 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

A. Write the time series in terms of W_t (∇Y_t).
B. What are the coefficients of this ARIMA process?

Part A: Rewrite the ARIMA process as

$Y_t - Y_{t-1} = 0.4Y_{t-1} + 0.1Y_{t-2} - 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

$= 0.4Y_{t-1} - 0.4Y_{t-2} + 0.4Y_{t-2} + 0.1Y_{t-2} - 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

$= 0.4Y_{t-1} - 0.4Y_{t-2} + 0.5Y_{t-2} - 0.5Y_{t-3} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

$\Rightarrow W_t = 0.4 W_{t-1} + 0.5W_{t-2} + e_t + 0.3e_{t-1} + 0.2e_{t-2}$

Part B: The ARIMA coefficients are

$\phi_1 = 0.4$

$\phi_2 = 0.5$
$\theta_1 = -0.3$
$\theta_2 = -0.2$

** Exercise 9.5: ARIMA Process

A time series is $Y_t = \theta_0 + \alpha_1 \times Y_{t-1} + \alpha_2 \times Y_{t-2} + \alpha_3 \times Y_{t-3} + \beta_1 \times e_t + \beta_2 \times e_{t-1} + \beta_3 \times e_{t-2}$

A. Write the time series in terms of W_t (∇Y_t).
B. What is the ARIMA process followed by this time series?

Part A: Rewrite the ARIMA process as

$Y_t - Y_{t-1} = \theta_0 + (\alpha_1 - 1) \times Y_{t-1} + \alpha_2 \times Y_{t-2} + \alpha_3 \times Y_{t-3} + \beta_1 \times e_t + \beta_2 \times e_{t-1} + \beta_3 \times e_{t-2}$

$Y_t - Y_{t-1} = \theta_0 + [(\alpha_1 - 1) \times Y_{t-1} - (\alpha_1 - 1) \times Y_{t-2}] + [(\alpha_1 - 1) \times Y_{t-2} + \alpha_2 \times Y_{t-2} + \alpha_3 \times Y_{t-3}] + \beta_1 \times e_t + \beta_2 \times e_{t-1} + \beta_3 \times e_{t-2}$

The coefficient of $Y_{t-1} - Y_{t-2}$ is $(\alpha_1 - 1)$.

For this to be an ARIMA(2,1,2) process, we must have

$$(\alpha_1 - 1) + \alpha_2 = -\alpha_3 \Rightarrow$$
$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

Jacob: What about the β coefficients?

Rachel: Any β coefficients are fine.

- If $\beta_1 = 1$, then $\phi_1 = -\beta_2$ and $\phi_2 = -\beta_3$.
- If $\beta_1 \neq 1$, then $\phi_1 = -\beta_2 / \beta_1$ and $\phi_2 = -\beta_3 / \beta_1$.

** Exercise 9.6: Non-Stationary Series

- A time series $Y_t = 2\ Y_{t-1} + \epsilon_t$ has $\sigma^2_\varepsilon = 3$.
- $Y_t = 0$ for $t < 1$.

A. What is the variance of Y_t for t = 1?
B. What is the variance of Y_t for t = 2?
C. What is the variance of Y_t for t = 3?
D. Is this time series stationary?

Part A: $Y_1 = 2\ Y_0 + \epsilon_1 = \epsilon_1$, so the variance of Y_1 = the variance of $\epsilon_t = \sigma^2_\varepsilon = 3$.

Part B: $Y_2 = 2\ Y_1 + \epsilon_2 = 4\ Y_0 + 2\ \epsilon_1 + \epsilon_3 = \epsilon_1$, so the variance of $Y_2 = 2^2 \times \sigma^2_\varepsilon + \sigma^2_\varepsilon = 15$.

Part C: The variance of Y_t is $\sigma^2_\varepsilon + 2^2\ \sigma^2_\varepsilon + (2^2)^2\ \sigma^2_\varepsilon + \ldots + (2^2)^{t-1}$

$= \sigma^2_\varepsilon \times (2^{2 \times t} - 1) / (2^2 - 1)$

$= ⅓ \times (4^t - 1) \times \sigma^2_\varepsilon$, which is equivalent to equation 5.1.4 on page 89.

The variance of Y_3 is $⅓ \times (64 - 1) \times 3 = 63$.

Part D: A stationary time series has the same mean and variance for all values of *t*. The variance of this time series depends on *t*.

Jacob: What if the exercise did not say that $Y_t = 0$ *for* $t < 1$. What the time series be stationary?

Rachel: A stationary time series need havd no beginning. It is in a stochastic equilibrium: the mean and variance are the same in all periods. If this time series has no beginning, its variance is infinite and its has no mean.

Jacob: Why does it have no mean? The mean of ϵ is zero, so isn't the mean of Y_t also zero?

Rachel: If $Y_j = k$, the expected value of Y_{j+1} is $2 \times k$, and the expected value of Y_{j+1} is $2 \times 2 \times k$. if *k* is zero, these are all zero. But Y_j has infinite variance, so *k* could be anything.

** Exercise 9.7: Combining error terms

Suppose $Y_t = M_t + e_t$ and $M_t = M_{t-1} + \epsilon_t$

A. Write Y_t as a function of M_{t-1} and error terms.
B. What type of time series is M_t?
C. What type of time series is Y_t?
D. What is ∇Y_t (the first difference of Y_t)?
E. What is the variance of Y_t?
F. What is the variance of ∇Y_t?
G. What is the covariance of ∇Y_t and ∇Y_{t-1}?
H. What is ρ_1, the autocorrelation of ∇Y_t and ∇Y_{t-1}?

Part A: $Y_t = M_t + e_t = M_{t-1} + e_t + \epsilon_t$

Part B: M_t is a random walk.

Part C: $Y_t = M_{t-1} + e_t + \epsilon_t = Y_{t-1} + \epsilon_t + e_t - e_{t-1}$. This is a random walk with a more complex error term.

Part D: $\nabla Y_t = Y_t - Y_{t-1} = M_{t-1} + e_t + \epsilon_t - (M_{t-1} + e_{t-1}) = \epsilon_t + e_t - e_{t-1}$

Part E: If the random walk has no beginning, the variance is infinite, so it does not exist. If the random walk has a beginning, the variance depends on the period.

Part F: The variance of ∇Y_t = var($\epsilon_t + e_t - e_{t-1}$). The three random variables are independent, so the variance = $2\sigma^2_e + \sigma^2_\varepsilon$.

Part G: The covariance of ∇Y_t and ∇Y_{t-1} is covariance ($\epsilon_t + e_t - e_{t-1}$, $\epsilon_{t-1} + e_{t-1} - e_{t-2}$) = $-\sigma^2_e$.

Part H: The autocorrelation of ∇Y_t and ∇Y_{t-1} (ρ_1) is $-\sigma^2_e / 2\sigma^2_e + \sigma^2_\varepsilon = -1 / (2 + \sigma^2_\varepsilon / \sigma^2_e)$. This is equation 5.1.10 on page 90.

** Exercise 9.8: IMA(1,1) process

Each of the following time series is an IMA(1,1) process. What is the value of θ for each time series?

A. $Y_t = Y_{t-1} + e_t - 0.4e_{t-1}$
B. $Y_t = Y_{t-1} - e_t - 0.4e_{t-1}$
C. $Y_t = Y_{t-1} + 0.4e_t - 0.4e_{t-1}$
D. $Y_t = Y_{t-1} - 0.4e_t - 0.4e_{t-1}$

Part A: The first difference of an IMA(1,1) is an MA(1) process.

The first difference of this time series is $e_t - 0.4e_{t-1}$, which is an MA(1) process with $\theta = 0.4$.

Part B: The first difference of this time series is $- e_t - 0.4e_{t-1}$. Use a change of the error term $\epsilon_{t'} = -\epsilon_t$, which gives a first difference of $+ e_{t'} + 0.4e_{t'-1}$, which is an MA(1) process with $\theta = -0.4$.

Part C: The first difference of this time series is $0.4\, e_t - 0.4e_{t-1}$. Use a change of the error term $\epsilon_{t'} = 2.5\epsilon_t$, which gives a first difference of $+ e_{t'} - e_{t'-1}$, which is an MA(1) process with $\theta = 1$.

Part D: The first difference of this time series is $-0.4\, e_t - 0.4e_{t-1}$. Use a change of the error term $\epsilon_{t'} = -2.5\epsilon_t$, which gives a first difference of $+ e_{t'} + e_{t'-1}$, which is an MA(1) process with $\theta = -1$.

(Cryer and Chan Page 93)

** Exercise 9.9: ARI(1,1) process

The time series $Y_t = \theta_0 + \alpha\ Y_{t-1} + \beta\ Y_{t-2} + e_t$ is an ARI(1,1) process.

A. Write the time series in terms of W_t (∇Y_t).
B. What is the relation of α and β?
C. What is the value of ϕ for this ARI(1,1) process?

Part A: $W_t = \nabla Y_t = Y_t - Y_{t-1} = \theta_0 + (\alpha - 1) \times Y_{t-1} + \beta\ Y_{t-2} + e_t$

Part B: If $\alpha - 1 = -\beta$, we can write the time series as $Y_t - Y_{t-1} = \theta_0 + (-\beta) \times (Y_{t-1} - Y_{t-2}) + e_t$

Part C: $\phi = -\beta = \alpha - 1$

** Exercise 9.10: Time series process

A time series is $Y_t = \theta_0 + 1.75\ Y_{t-1} - 0.75Y_{t-2} + e_t$ is an ARI(1,1) process.

A. Write the time series in terms of W_t (∇Y_t).
B. What is the value of ϕ_1 for this ARI(1,1) process?
C. What is the value of ϕ_2 for this ARI(1,1) process?

Part A: $W_t = \nabla Y_t = Y_t - Y_{t-1} = \theta_0 + (\alpha - 1) \times Y_{t-1} + \beta\ Y_{t-2} + e_t$

Part B: If $\alpha - 1 = -\beta$, we can write the time series as $Y_t - Y_{t-1} = \theta_0 + (-\beta) \times (Y_{t-1} - Y_{t-2}) + e_t$

Part C: $\phi = -\ \beta$

** Exercise 9.11: IMA(1,1) process

An IMA(1,1) process is $Y_t = Y_{t-1} + \epsilon_t - 0.4\ \epsilon_{t-1}$, with $Y_t = 0$ for $t < 1$.

A. Write Y_t as $\beta_t \times \epsilon_t + \beta_{t-1} \times \epsilon_{t-1} + \ldots + \beta_1 \times \epsilon_1 + \beta_0 \times \epsilon_0$
B. What is β_t?
C. What is β_{t-1}?
D. What is β_1?
E. What is β_0?

Part A: Expand the time series period by period:

$Y_t = Y_{t-1} + \epsilon_t - 0.4\ \epsilon_{t-1}$

$Y_{t-1} = Y_{t-2} + \epsilon_{t-1} - 0.4\ \epsilon_{t-2}$

$Y_{t-2} = Y_{t-3} + \epsilon_{t-2} - 0.4\ \epsilon_{t-3}$

The expanded time series is

$Y_t = \epsilon_t - 0.4\ \epsilon_{t-1} + \epsilon_{t-1} - 0.4\ \epsilon_{t-2} + \epsilon_{t-2} - 0.4\ \epsilon_{t-3} + \ldots + Y_0 + \epsilon_1 - 0.4\ \epsilon_0$

$Y_0 = 0$, so we have finished expanding. We group error terms with the same subscript to get the β_t values.

Part B: β_t is the coefficient of the ϵ_t term = 1.

Part C: β_{t-1} is the coefficient of the ϵ_{t-1} term = (1 – 0.4).

Part D: β_1 is the coefficient of the ϵ_1 term = (1 – 0.4).

Part E: β_0 is the coefficient of the ϵ_0 term = –0.4.