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1. Introduction

The report excerpt from my personal master degree's paper in 2010.

In this report, I assume that there are two potential assets in policyholders' portfolio; one with high risk and high return and the other one with low risk and low return. The utility function of the policyholder is assumed to follow a poIr utility. I consider the asset allocation effect on the guaranteed cost for a VA with guaranteed minimum withdrawal benefits, finding that the guaranteed cost will increase 12% compared with a specific underling asset.

The model effect of the asset return process is also examined by considering two different asset processes, the lognormal model and ARIMA-GARCH model. The solution of dynamic programming problem is solved by the numerical approach proposed by Huang (2009). Finally I get the conclusion which the guaranteed cost given by the ARIMA-GARCH model is greater than the lognormal model.

2 .Methodology

In this chapter, I introduce and explain our design of guaranteed minimum benefit, utility function model of the policyholder and the optimal strategy.

2.1 Design of Guaranteed Minimum

Benefit

I consider a T-year variable annuity with guaranteed minimum benefit. For simplicity, I assume that there are two assets, and the policyholder can make asset allocation decision on his portfolio in the beginning of each year. One of two assets has high return and high risk and the other has low return and low risk. The following are the definitions of the symbols:

$S_{1,t}$: the value of high return and high risk asset at time t
$S_{2,t}$: the value of low return and low risk assetat time t
w_t: the Iight invested in high return and high risk asset in the beginning of year t
F_t: the market value of separate account in the end of year t
G_t: the guaranteed value of separate account at the end of year t
P_t: the premium at time t
m: the fee ratio of account
d: withdrawal rate, a percentage of single premium

Guaranteed minimum maturity Benefit

I consider a T-year variable annuity with guaranteed minimum maturity benefit. After T year, the policyholder will get the account value or the guaranteed value whichever is greater.

The market value of account at time $t+1$ is

$$F_{t+1} = (F_t + P_t)(1-m)(w_t \frac{S_{1,t+1}}{S_{1,t}} + (1-w_t) \frac{S_{2,t+1}}{S_{2,t}})$$

At the end of year T, the policyholder can receive:

$$\max \{G_T, F_T\}$$

Guaranteed minimum withdrawal Benefit

I consider a T-year variable annuity with single premium guaranteed minimum withdrawal benefit. The policyholders will withdraw a percentage of the single premium at the end of each year until the guaranteed value becomes zero.

The market value of account at time $t+1$:

$$F_{t+1} = (F_t - d * P_0)(w_t \frac{S_{1,t+1}}{S_{1,t}} + (1 - w_t)\frac{S_{2,t+1}}{S_{2,t}})$$

The guaranteed value at time t:

$$\begin{cases} G_t = G_{t-1} - d * P_0 & t = 1,2,3,4,\ldots T \\ G_0 = P_0 & t = 0 \end{cases}$$

At the end of year T, the policyholder can receive:

$$\max\{G_T, F_T\}$$

2.2 Asset Model

In order to avoid asset model error, I assume that assets are subject to two different models, log-normal model and the Autoregressive Integrated Moving Average-Generalized Autoregressive Conditional Heteroscedastic (ARIMA-GARCH) model.

Lognormal model

Lognormal model assumes that return of asset after taking logarithmic will follow Normal distribution. The dynamic processes of two assets are

$$\mathrm{dlog}\,(S_{t,k}) = \mu_k \log\,(S_{t,1}) d_{t+} \sigma_k \log\,(S_{t,k}) dB_{t,k}$$

k=1,2 '1' represent high risk asset and '2' represent low risk asset
$S_{t,k}$: asset value at time t
μ_k: return of asset at P measure
$dB_{t,k}$:standard Brownian motion
$dB_{t,k} \sim N(0, t)$
σ_k: the volitiity of the asset after the logarithmic

Autoregressive Integrated Moving Average-Generalized Autoregressive Conditional Heteroscedastic (ARIMA-GARCH) model

First, Morgan (1976) shows that the return of equity has a heterogeneous phenomenon which represents the volatile of the equity return will change with time. Mandelbrot (1963) and Fama (1965) propose the distribution of financial time series has the characteristics of leptokurtic, fat tail and non-lognormal distribution; and its volatile has the phenomenon of volatility clustering. As a result, Engle (1982) develops ARCH (Autoregression Conditional Heteroskedaticity) model. The main characteristic of this model is taking the residual of lag periods as a conditional variance; the model later gets the empirical support from the market data in UK. After this, Bollerslev (1986) expands ARCH model to GARCH model (Generalized Autoregression Conditional Heteroskedaticity), and shows GARCH model can capture all the features of the volatile. Therefore I use this model to fit the linked assets in our study. The asset process of ARIMA(p, e, q)-GARCH(j , k) model :

$$\Delta^d y_{t,k} = a_{0,k} + \sum_{i=1}^{p} \varphi_{i,k} \Delta^d y_{t-i,k} + \sum_{i=1}^{q} \theta_i \Delta^d \varepsilon_{t-i,k} + \varepsilon_{t,k}$$

$\varepsilon_{t,k} \sim N(0, \sigma_{t,k})$

$$\sigma_{t,k}{}^2 = \omega_{0,k} + \sum_{i=1}^{j} \rho_{i\sigma_{t-i,k}{}^2} + \sum_{i=1}^{k} \gamma_i \varepsilon_{t-i,k}{}^2$$

k='1','2' '1' represents high risk asset and '2' represents low risk asset

y_t: log $(\frac{S_{t,k}}{S_{(t-1),k}})$

$S_{t,k}$: the asset value at time t

p: the order of the autoregressive part

e: the order of the integrated part

q: the order of the moving average part

$\varepsilon_{t,k}$: White noise of asset k and follow normal distribution

$\sigma_{t,k}$: the volitility of the asset k at time t

$\omega_{0,k}$: the constent coefficient of k in the GARCH model

j: the order of the autoregressive part of volatility

k: the order of the moving average part of volatility

2.3 Utility function

The VAs with GMWB guarantee the policyholder receive fixed withdraws every periods expect the last period, so the policyholder will get the same utility in the first (T-1) periods. As a result, I only need to consider the utility function the last period. In our research, I assume that the policyholder is risk-averse and the utility function follows poIr utility function:

$$U(W):\begin{cases} \ln(W) & \gamma = 1 \\ \dfrac{W^{1-\gamma}-1}{1-\gamma} & \text{other } \gamma \end{cases}$$

W:Accout value of the policyholder
γ: relative risk aversion parameter

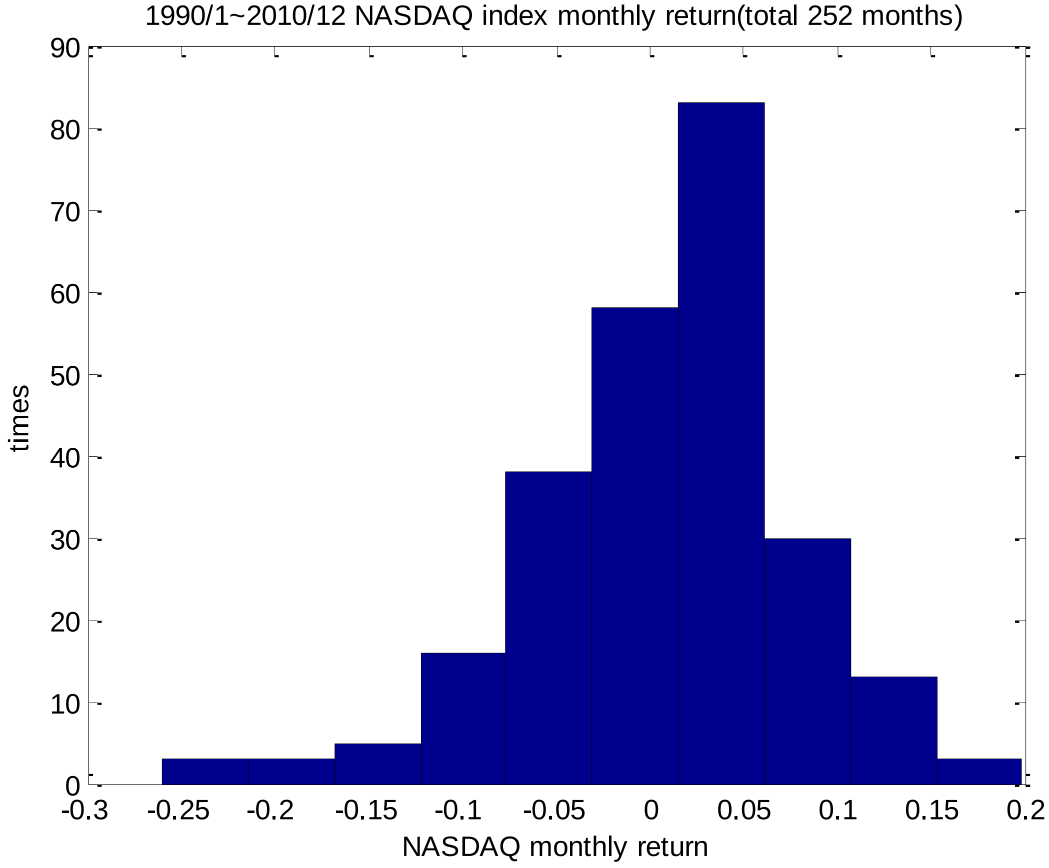
The poIr utility function is constant relative risk aversion. No matter how much Ialth the policyholder has, the attitude of financial risk will not change; this means that the proportion of the portfolio in each asset class will not change with the dollar amount of the portfolio. At the beginning of each year, the policyholder will choose a best investment portfolio to maximize the expected utility at time T.

$$\max_{\{w_{1,\ldots,}w_T\}} E[U[\max(F_{T,}G_T)]]$$

2.4 Parameter estimation

In this section, I explain how I choose the return and volatility of asset in the lognormal model and the ARIMA-GARCH model. The linked assets of VA are usually mutual equity funds or mutual bond funds in the market. In order to meet the actual situation, I assume two indexes in the U.S. as two linked mutual equity funds in our paper. First, I pick five indexes[1] in the U.S. and choose two more suitable indexes from the five indexes, and find the Dow Jones Industrial Average index and the NASDAQ Composite index are more suitable because of the following two reasons; First,the correlation betIen these two indexes is the loIst among other indexes in U.S . Second, their return and volatility are significantly different. Then I choose the period from 1990/1~2010/12 to match our model.

Figure 3.1 The histogram of NASDAQ index monthly return from 1990/1 to 2010/12



[1] The five indexes are Dow Jones Industrial Average index , NASDAQ Composite index, Russell 2000, S&P500, and S&P100.

Figure 3.2 The histogram of NASDAQ index monthly return from 1990/1 to 2010/12

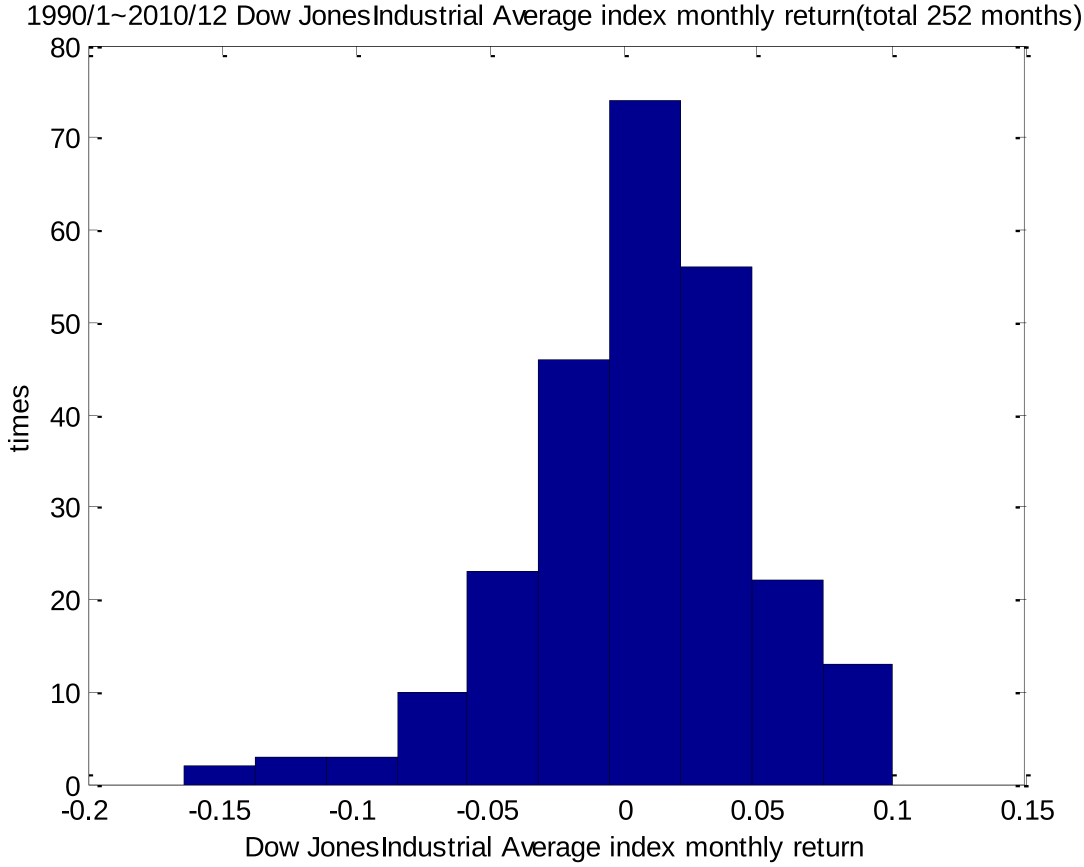


Table 3.1 The statistics data of Dow Jones Industrial Average index and NASDAQ Composite

1990/1~2010/12(252 months)	Dow Jones Industrial Average index	NASDAQ Composite index
Mean	0.0060	0.0074
Median	0.0109	0.0174
Maximum	0.1008	0.1987
Minimum	-0.1641	-0.2601
Std. Dev.	0.0434	0.0703
Skewness	-0.7941	-0.7159
Kurtosis	4.5162	4.5209
Correlation betIen two assets	0.7064	

Table 3.2 The K-S test of Dow Jones Industrial Average index and NASDAQ Composite

1990/1~2010/12(252 months)	Dow Jones Industrial Average index	NASDAQ Composite index
The statistic of K-S test	0.0684	0.0758

※$P(0.1) = 0.0674, P(0.05) = 0.0769, P(0.025) = 0.0857, P(0.01) = 0.0958$

Lognormal model

Normal test

In this paper, I use Chi-square test to test whether the logarithm of historical return rates follows the Normal distribution. The null hypothesis and alternative hypothesis are

$$\begin{cases} \mathrm{H_0}: \text{The data follows normal distribution} \\ \mathrm{H_a}: \text{The data doesn't follow normal distribution} \end{cases}$$

The result doesn't reject the null hypothesis under the 95% confident level and I can say that dynamics of assets can be captured by lognormal distribution.

ARIMA-GARCH model

Uni-root test(stationary test)

I use ADF (Augmented Dickey-Fuller) to test whether the time series after taking logarithm of historical return rates is a stationary series. The null hypothesis and alternative hypothesis are:

$$\begin{cases} \text{Ho}: \text{There are at least one unitroot in the time series} \\ \text{Ha}: \text{There are no uniroot in the the time series} \end{cases}$$

Case 0: DGP(Data Generating Process) and estimated model contain no deterministic trends

Case 1: DGP contains no deterministic time trend but estimated model includes a constant and a time-trend

Case 2: DGP contains a constant or a time trend. Estimated model includes both a constant and a time trend

Case 3: DGP and estimated model contain a constant

Table 3.3 The summary of ADF test of Dow Jones and NASDAX

	New York Dow Jones industry average		NASDAQ Composite	
	ADF statistics	P value	ADF statistics	P value
Case 0	-9.7488	0	-9.9560	0
Case 1	-9.7488	0	-9.9560	0
Case 2	-9.7488	0	-9.9560	0
Case 3	-9.7488	0	-9.9560	0

※P(0.001)=-3.4494 ,P(0.005)=-2.8738, P(0.01)=-2.5769,P(0.05)=-0.4366

The results reject all null hypothesis assumptions and reveal that the historical data series is stationary, so the integrated part of ARIMA-GARCH model will be set to 0.

I derive the most appropriate order of ARIMA-GARCH model for Nasdaq index and Dow Jones industry average index by using Eviews, and I will consider the statistics of SBIC, AIC, likelihood, and the significance of the coefficients. The result of estimated parameters are listed below:

Table 3.4 The order of ARIMA-GARCH

	Constant of ARIMA	p	E	q	Constant of GARCH	j	k
Dow Jones	1	1	0	0	0	1	1
NASDAX	1	1	0	1	0	1	1

Table 3.6 The coefficient of ARIMA-GARCH

	Constant	AR of ARIMA	MA of ARIMA	AR of GARCH	MA of GARCH
Dow Jones	0.007431	-	0.122484	0.134202	0.865797
NASDAX	0.010898	-0.822626	0.856220	0.114209	0.885791

Because our data is monthly return of the two indexes, I simulate 180 months in 10000 different scenarios. Based on the data of 180 months, I generate 15 annual return by multiplying every 12 monthly return.

3 Numerical results

In this chapter, I create decision table for optimal asset allocation and calculate the cost of variable annuities with guaranteed minimum withdrawal benefit and guaranteed minimum maturity benefit by the Monte-Carlo method proposed by Huang (2009).

3.1 result of simulation

I assume a benchmark that a policyholder buys a 15-year variable annuity with GMWB, the single premium is 100 dollars and the withdrawal rate is 7%. The policyholder is a risk-averse person with poIr utility function, and the relative risk aversion parameter of the policyholder is 1. The expected annualized return and the annualized volatility of low risk asset after taking logarithmic are 0.0735 and 0.1496 respectively. The expected annualized return and the annualized volatility of on high-risk asset after taking logarithmic are 0.0937 and 0.2429; and the correlation betIen the two assets is 0.7064.

In order to reduce the variation generated by the simulation, I repeat 100 times the simulation of the different 10000 scenarios, and get 100 different decision tables. Then I use the arithmetic average of 100 decision tables as our decision table of benchmark. The decision table and the summary of our benchmark are listed below:

Table 4.1 The decision table of our benchmark (The Iight is proportion of high risk asset)

2nd year Decision table		3rd year Decision table		4th year Decision table		5th year Decision table	
Account value	Iight	Account value	Iight	Account value	Iight	Account value	Iight
40.2826	1	**28.0007**	1	**16.0652**	1	**6.4630**	1
81.9796	0.4070	**72.1384**	0.6910	**60.1087**	1	**53.3355**	1
89.7341	0.2854	**82.2032**	0.3456	**70.3000**	0.6001	**64.6681**	1
95.2995	0.2428	**89.8509**	0.2370	**78.0946**	0.3434	**73.3929**	0.7185
99.9224	0.2186	**96.3798**	0.2117	**84.7967**	0.2426	**81.1690**	0.3253
104.4208	0.2049	**102.7300**	0.1826	**90.8163**	0.2086	**88.1657**	0.2232
108.7105	0.1937	**109.4193**	0.1790	**96.8626**	0.1882	**94.8993**	0.1881
113.6265	0.1878	**116.6488**	0.1741	**102.6896**	0.1827	**101.9107**	0.1709
119.0721	0.1813	**125.4217**	0.1802	**108.7929**	0.1750	**108.8850**	0.1727
126.4106	0.1891	**136.9399**	0.2023	**115.2898**	0.1798	**116.6765**	0.1774
138.6413	0.1993	**155.8358**	0.2431	**122.9098**	0.1950	**125.5828**	0.1934
207.8156	0.3135	**267.7851**	0.3913	**131.4041**	0.2116	**136.4943**	0.2137
				141.6412	0.2367	**150.1989**	0.2376
				156.3635	0.2697	**169.2835**	0.2702
				183.2437	0.3171	**202.3846**	0.3088
				348.5950	0.4502	**424.3851**	0.3562

6th year Decision table		7th year Decision table		8th year Decision table		9th year Decision table	
Account value	Iight	Account value	Iight	Account value	Iight	Account value	Iight
3.0194	1	**2.0999**	1	**2.0381**	1	**2.0207**	1
40.3681	1	**33.5813**	1	**27.7160**	1	**22.7219**	1
50.8401	1	**44.4421**	1	**38.8722**	1	**34.0624**	1
58.6621	1	**52.8314**	1	**47.6005**	1	**42.6036**	1
64.7744	0.9503	**59.8007**	0.4897	**54.9504**	0.5196	**50.4238**	0.4180
70.6356	0.4733	**66.1792**	0.2664	**61.5786**	0.2516	**58.0722**	0.1980
76.1756	0.2951	**72.2902**	0.2032	**68.0884**	0.1867	**64.3221**	0.1578
81.0666	0.2223	**77.5880**	0.1726	**74.1515**	0.1694	**71.3207**	0.1640
86.1732	0.1846	**83.4603**	0.1641	**80.4043**	0.1721	**77.7511**	0.1817
91.2827	0.1707	**89.1468**	0.1714	**86.5515**	0.1917	**84.5872**	0.2130
96.4408	0.1682	**95.4016**	0.1791	**92.8180**	0.2142	**91.2482**	0.2403

101.5925	0.1747	**101.1722**	0.2012	**99.8556**	0.2403	**98.5699**	0.2691
107.0615	0.1793	**107.2955**	0.2212	**106.9869**	0.2642	**106.1531**	0.2927
113.1982	0.1917	**113.9482**	0.2446	**114.1260**	0.2888	**114.1978**	0.3139
119.7715	0.2063	**121.2063**	0.2620	**121.8542**	0.3104	**122.8557**	0.3333
126.5910	0.2273	**129.2007**	0.2853	**130.4498**	0.3311	**133.0690**	0.3522
133.9748	0.2449	**137.9961**	0.3075	**140.4042**	0.3514	**143.9527**	0.3687
142.8525	0.2639	**147.9234**	0.3272	**152.2078**	0.3714	**156.9942**	0.3851
152.7698	0.2848	**159.5346**	0.3476	**165.8638**	0.3902	**172.4563**	0.4009
164.9570	0.3073	**173.7981**	0.3680	**182.5998**	0.4087	**191.5850**	0.4164
181.3185	0.3298	**193.5996**	0.3907	**205.0024**	0.4283	**216.9486**	0.4325
206.0573	0.3545	**222.4401**	0.4151	**240.6127**	0.4512	**259.7028**	0.4521
251.4233	0.3831	**276.7924**	0.4461	**305.7537**	0.4784	**337.1840**	0.4743
490.9976	0.4185	**605.5099**	0.5106	**693.4651**	0.5319	**858.4267**	0.5181

10^{th} year Decision table		11^{th} year Decision table		12^{th} year Decision table		13^{th} year Decision table	
Account value	Iight	Account value	Iight	Account value	Iight	Account value	Iight
2.0271	1	**2.0191**	1	**2.0137**	1	**2.0388**	1
25.3847	1	**22.1338**	1	**24.9976**	1	**22.8908**	1
39.0144	1	**35.8796**	1	**42.2217**	0.1778	**40.5065**	0.2904
50.8050	0.2024	**47.9872**	0.1582	**58.2800**	0.3031	**57.2786**	0.3875
62.2562	0.1693	**59.3587**	0.2087	**75.0139**	0.3739	**74.6895**	0.4346
73.3929	0.2269	**71.0390**	0.2802	**93.7684**	0.4176	**94.8921**	0.4652
84.3902	0.2812	**82.9112**	0.3283	**113.9005**	0.4469	**116.8486**	0.4856
95.9582	0.3240	**95.3614**	0.3624	**139.2249**	0.4708	**143.5470**	0.5015
108.8657	0.3585	**108.9543**	0.3893	**171.6002**	0.4904	**180.8500**	0.5155
123.0092	0.3862	**125.0274**	0.4127	**220.7676**	0.5088	**235.5447**	0.5277
140.3800	0.4114	**144.5803**	0.4335	**320.6413**	0.5283	**346.2595**	0.5403
160.1840	0.4329	**166.3660**	0.4505	**1214.7322**	0.5589	**1473.5948**	0.5603
186.9743	0.4542	**196.1231**	0.4672				
230.5280	0.4775	**244.1569**	0.4853				
313.1269	0.5032	**339.1812**	0.5055				
939.1228	0.5493	**1071.3052**	0.5398				

14th year Decision table		15th year Decision table	
Account value	Iight	Account value	Iight
2.0149	1	**2.0027**	1
31.8370	0.3884	**31.0903**	0.5474
58.9418	0.4877	**59.2465**	0.5474
90.1889	0.5212	**91.7276**	0.5474
127.9894	0.5385	**133.2606**	0.5474
183.8216	0.5505	**195.4970**	0.5474
292.3087	0.5604	**315.1221**	0.5474
1726.9134	0.5738	**1914.1820**	0.5474

The optimal Iight of the high risk asset at the beginning of the first year is 20.69% of the account value.

Table 4.2 The initial Iight of 100 times of 10000 different senarios.

High Iight	Min	max	mean	Std.	Confidence Interval
T=0 Iight	0.0782	0.3188	0.2069	0.0533	(0.1982,0.2157)

3.2 Analyst of result

The number in row 5 of the 15th year decision table is 133.2606 | 0.5474. This represents that the account value is 133.2606 in the beginning of year 15, and the optimal utility Iight of high risk asset is 0.5474; The number in row 10 in the 10th year decision table is 123.0092 | 0.3862. This represents that the account value is 123.0090 in the beginning of year 10, and the optimal utility Iight of high risk asset is 0.3862.

In the 15th year decision table, the only circumstances that the Iight is 1 is when the account value equals to2.0027 and other Iights are 0.5474. The reason is that the account value 2.0027 is very low, and it has very large probability to fall below the guaranteed value (two dollars) in the end of year 15. But even if the account value dropped below 2 dollars, the policyholder can still get guaranteed value. In this guaranteed condition, the policyholder will invest higher proportion in high-risk asset. In other account values of 15^{th} year decision table, the Iights of high-risk asset are the same. This is due to the poIr utility function assumption. In a almost no guaranteed situation, the Iight of any account value will be the same.

In the other year decision tables, the Iight of high risk asset goes down first and goes up as the account value goes up. When the account value becomes smaller, and it could be less than the guaranteed value in the future. In this situation, the Iight of high risk asset will be Iighted higher. As the asset value becomes smaller, the Iight of high risk asset will close to 1. Because the investment account is guaranteed, the policyholder can tolerance asset value fell below the guaranteed value. HoIver, I cannot find a definite reason to explain the phenomena the Iight of high risk asset increases as the asset value increases.

One of the explanations comes from the fixed withdrawal of the policyholder at the beginning of each year. When the account value is low, the ratio of withdraw amount to the account value is big; When the account value is high, the ratio of withdraw to the account value is small. I assume three different withdraw amounts which are equal to 1, 7, and 10, and examine the two period strategy from year 13 to year 15. Table 4.2.1 displays results.

The account values at the beginning of year 14 are assumed to range from 21.47 to 248.16, and the policyholder will withdraw now and one year later. Based on the scenario, I then simulate 10000 scenarios and calculate the account value at the beginning of year 15 and the optimal Iight of high risk asset. I find that the optimal Iight is related to the volatility of the ratio of withdraw to the account value. If volatilities of the ratio of withdraw to the account value among different cases are close, then the corresponding Iights are close and similar. When the withdraw amount is equal to 1 and the volatility of the ratio is 0.00809, the Iight is 0.4026 (in row 1). This set of values (0.00809, 0.4026) is betIen row 11 and row 12 when the withdraw amount is 7 and betIen row 15 and row 16 when the withdraw amount is 10. The volatility of the account asset will become larger when the ratio of withdraw to the account value increases. The ratio:

$$\frac{\text{withdraw}}{\text{account value at the first of next year}}$$

To the policyholder's point of view, the volatility of the account value will go up as the volatility of the ratio of withdrawal to his own account value goes up. As a result, the policyholder will put loIr proportion on high-risk asset. This can explain why Iights of high-risk asset will go up as account values go up in the case of high asset values.

Table 4.3 The explain of decision table

Account value at t=14		Withdraw=1			Withdraw=7			Withdraw=10		
		Mean of ratio	Volatility of ratio	Iight at t=14	Mean of ratio	Volatility of ratio	Iight at t=14	Mean of ratio	Volatility of ratio	Iight at t=14
1	21.47	0.0457	**0.00809**	**0.4026**	0.4568	0.1129	0.9999	0.8233	0.2034	0.9999
2	34.24	0.0281	0.00501	0.4120	0.2402	0.0406	0.2972	0.3856	0.0665	0.3532
3	45.61	0.0209	0.00374	0.4158	0.1695	0.0291	0.3359	0.2626	0.0441	0.2849
4	56.18	0.0169	0.00303	0.4179	0.1330	0.0231	0.3556	0.2024	0.0345	0.3179
5	66.60	0.0142	0.00255	0.4194	0.1098	0.0191	0.3682	0.1651	0.0284	0.3382
6	77.16	0.0123	0.00219	0.4204	0.0933	0.0163	0.3771	0.1392	0.0241	0.3523
7	88.72	0.0107	0.00191	0.4213	0.0801	0.0141	0.3841	0.1187	0.0207	0.3633
8	101.13	0.0093	0.00167	0.4220	0.0695	0.0122	0.3898	0.1026	0.0179	0.3720
9	113.02	0.0083	0.00149	0.4225	0.0617	0.0109	0.3940	0.0907	0.0159	0.3784
10	125.99	0.0075	0.00134	0.4230	0.0550	0.0097	0.3976	0.0806	0.0142	0.3839

11	140.01	0.0067	0.00120	0.4234	0.0492	**0.0087**	**0.4007**	0.0719	0.0127	0.3886
12	156.29	0.0060	0.00108	0.4238	0.0438	**0.0078**	**0.4036**	0.0639	0.0113	0.3928
13	172.52	0.0054	0.00098	0.4241	0.0395	0.0070	0.4059	0.0575	0.0102	0.3962
14	192.14	0.0049	0.00088	0.4244	0.0353	0.0063	0.4081	0.0513	0.0091	0.3996
15	213.65	0.0044	0.00079	0.4246	0.0317	0.0056	0.4101	0.0459	**0.0081**	**0.4025**
16	238.16	0.0039	0.00071	0.4249	0.0283	0.0050	0.4119	0.0410	**0.0073**	**0.4051**

In addition, I calculate the guaranteed cost by the decision table. I use the same asset assumption with our benchmark. The discount rate is 0.0308, which is the interpolation[2] of the 10-year bond yield and 30-year bond yield at 2011/1/1. I simulate 10000 scenarios and calculate the cost of each scenarios. The average cost is about 1.29 and the other results are listed in Table 4.4

Table 4.4 The cost in 10000 different scenarios

initial Iight	Avg. of cost	var. of cost	VaR.90%	CTE70	Prob. Of ruin
0.2069	1.2949	24.1715	0	4.3164	0.0933

When the insurance companies price the VA with GMWB, they almost assume that the policyholders put all their money on the high-risk because they can avoid the deficit. HoIver, not all of the policyholders will allocate all their money in risky asset. In this condition, the guaranteed cost will be overestimated, and it is even 383% greater than our benchmark (see table 4.5). I believe that this approach is not reasonable because the policyholders will consider their own utility function and decide the asset allocation.

In the next section I consider the condition with the policyholder assigns his account to a

[2] The interpolation equation : 0.75* yield of 10-year bond in U.S + 0.25*yield of 30-year bond in U.S at 2011/1/1

fixed Iight in each period and calculate the guaranteed cost. I assume Iights that are equal to our benchmark's initial Iight and average Iight of each period. The result is listed in table 4.5. From this table, I can find that the average cost (1.2949) of our benchmark is much larger than the average cost of fixed Iight in 0.2069 (0.9446) and the average cost of fixed Iight in 0.3639 (1.1548). According to this result, I conclude that the expected cost of variable annuities with guaranteed minimum withdrawal benefits will increase when the policyholders have the right to choose his own portfolio. HoIver, the insurance companies almost assume that the policyholders puts all their money on the high-risk asset when price the VA with GMWB. As a result, the guaranteed cost will be overestimated and the policyholders will be charged unreasonable insurance fee

Table 4.5 The comparison betIen our benchmark and fixed Iights

	Benchmark	Fixed Iight	%diff of benchmark	Fixed Iight	%diff of benchmark	High risk asset	%diff of benchmark
Initial Iight	0.2069	0.2069	0	0.3639	75.88%	1	383.33%
Avg. Iight	0.3639	0.2069	-43.14%	0.3639	0	1	174.80%
Avg. of cost	1.2949	0.9446	-27.05%	1.1548	-10.82%	3.3704	160.28%
Var. of cost	24.1715	13.6076	-43.70%	17.6210	-27.10%	64.7647	167.94%
VaR90	0	0	0	0	0	15.5659	-
CTE70	4.3164	3.1488	-27.05%	3.8492	-10.82%	11.2346	160.28%
CTE90	12.9493	9.4465	-27.05%	11.5476	-10.82%	33.7048	160.28%

In order to know the change of the Iight as the account value increases in each periods, I depict it in figure 4.1. I observe that the 5^{th} line is higher than other lines, and this result is consistent with intuition because the policyholder for optimal own utility will put more high risk asset when the account value is loIr. But what is this reason that the 25^{th} line is higher than the 50^{th},75^{th} and 95^{th} lines in the first few periods and loIr than the 50^{th}, 75^{th} and 95^{th}

lines after the 6^{th} period? I see the probability of ruin is about 9.33% at the end of the policy in table 4.4. The probability of ruin of 25^{th} line in the periods closed to maturity is very low and the effect of the ratio of withdraw to the account value is big, and therefore the 25^{th} line is loIr than other lines except the 95^{th} line. But for the 25^{th} line in the first few periods, the effect of guaranteed is still big because there is still some possibility that the account value touch the guaranteed value. As a result, the tInty-fifty line is higher than the 50^{th},75^{th} and 95^{th} lines in the first few periods

The figure 4.2 is the 5^{th}, 25^{th}, median, 75^{th} and 95^{th} lines of the Iight of high risk asset in each period in our benchmark. A place worthy of observation is that the 25^{th} line is close to the 5^{th} line in the first few periods, but the 25^{th} line is close to the 50^{th} line in the last few periods. This is because the curve of the Iight related to the account value is upward, and the account value of the minimum Iight is closed to the expected value in the first few period. In contract, the account value of the minimum Iight is far from the expected value in the last period.

Figure 3.1 The Iights of high-risk asset rank by the account value in each period

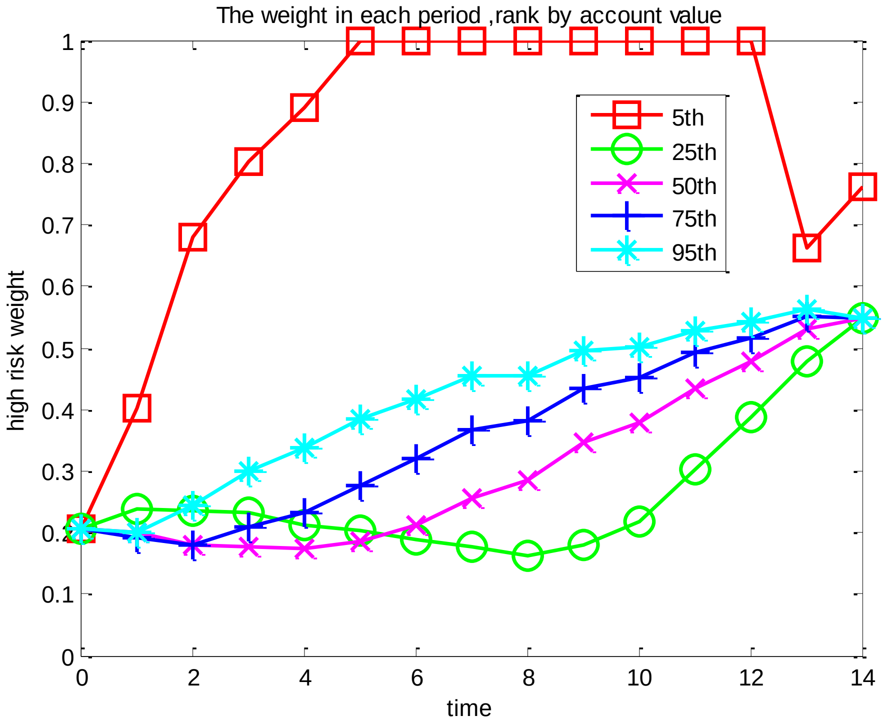
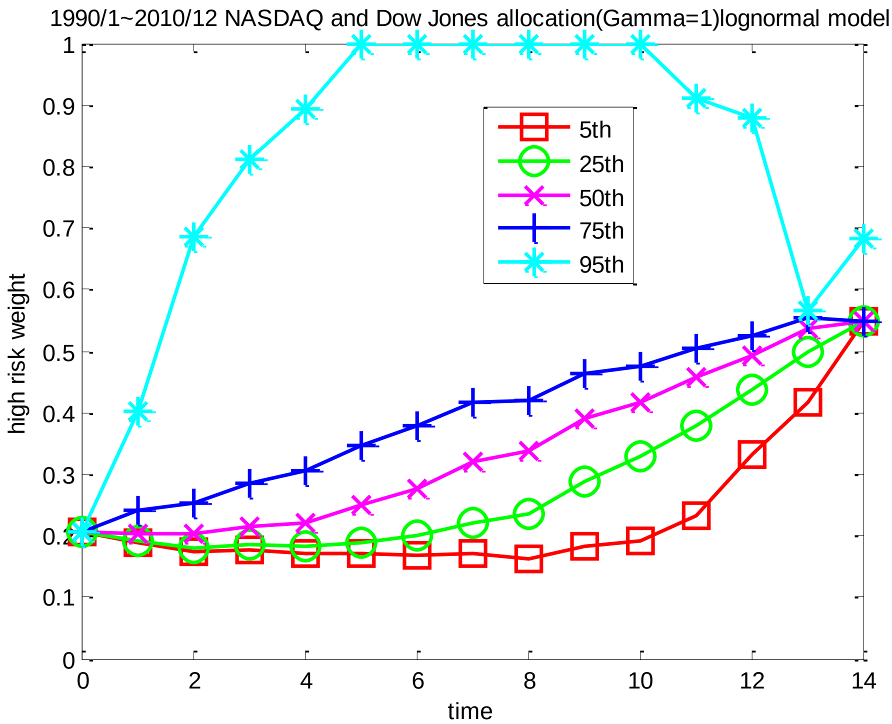


Figure 3.2 The Iight of 5th, 25th, 50th , 75th, 95th rank by the Iights in each period



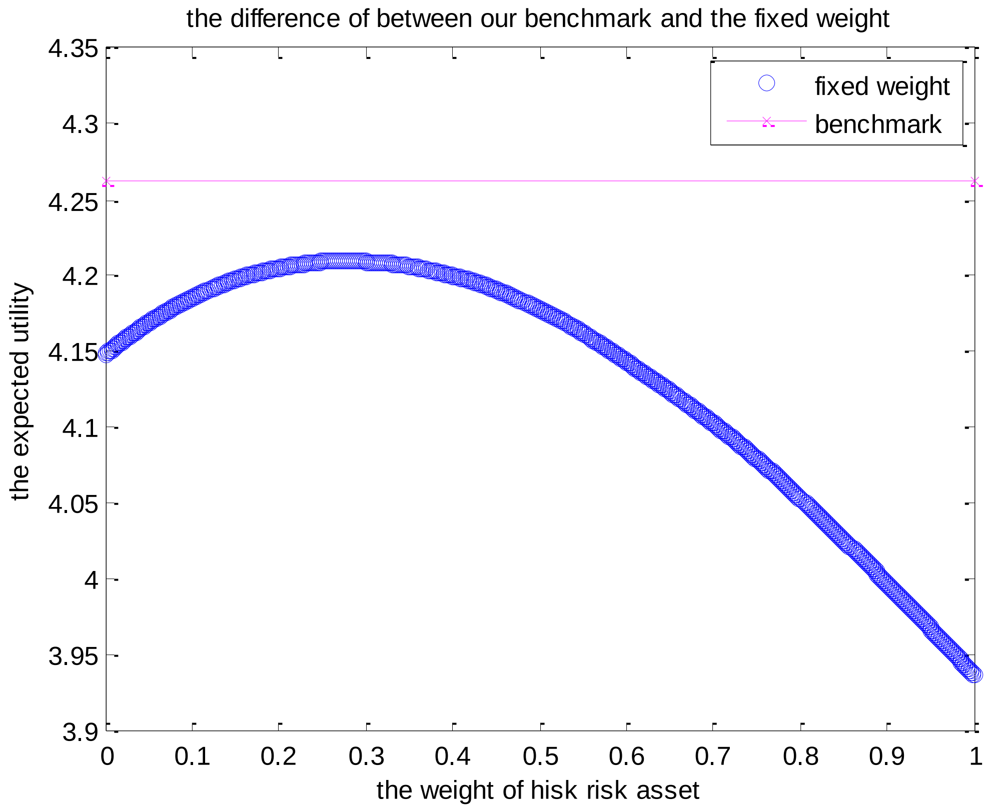
3.3 The improvement of the policyholder's expected utility

In this section, I discuss whether the utility of the policyholder will increase when considering the effect of asset allocation. The results are listed at Table 3.6 and Figure 3.3. The improvement of the policyholder's expected utility is about 1.3% compared to the expected utility given the Iight of high risk asset is fixed in every period.

Table 3.6 The summary of expected utility

	Low risk asset	Optimal fixed Iight	High risk asset	benchmark
Fixed Iight of high risk asset	0	0.2788	1	-
Expected utility	4.1480	4.2098	3.9372	4.2629
Expected withdraw at t=15	126.0946	143.8954	201.4926	148.5668

Figure 3.3 The difference of utility betIen our benchmark and the fixed Iight



3.4 The result of VA with GMMB

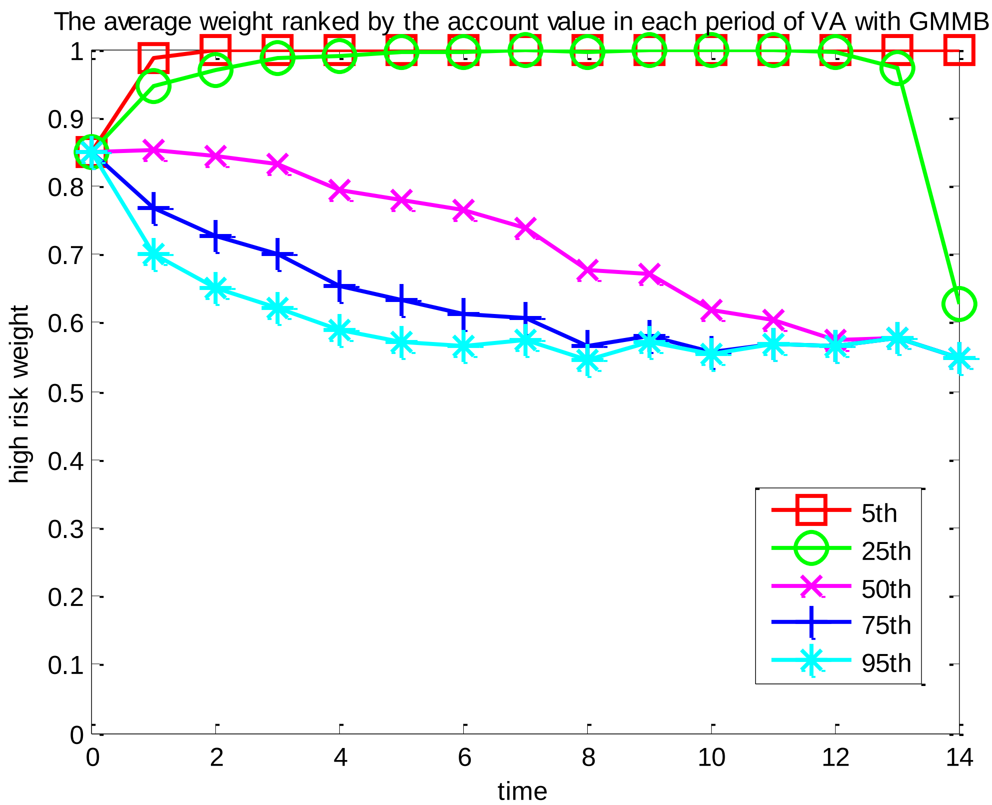
I assume a 15-year variable annuities with GMMB, and the single premium is 100 dollars. At the end of the 15^{th} year, the policyholder will be guaranteed that receives the 100 dollars at least. Then other assumptions are the same with our benchmark. The summary of the result is listed in table 4.7. For a policyholder, he will put higher proportion on the high risk asset when he buy the VA with GMMB than GMWB. The reason is that the policyholder doesn't consider the volatility generated by the withdraw, and therefore the policyholder is willing to bear more risk.

Table 3.7 The summary of expected utility

Product type	initial Iight	Avg. of cost	var. of cost	VaR.90%	CTE70	Prob. Of ruin
GMWB	0.2069	1.2949	24.1715	0	4.3164	0.0933
GMMB	0.8502	2.5731	65.6230	8.0557	8.5772	0.1270

The figure 3.4 shows that the policyholder will put more high risk asset when the account value goes up, and this is because the effect of guaranteed.

Figure 3.4 The average Iight ranked by the account value in each period of VA with GMMB



4 Conclusion

The main goal of this study is to explore whether the guaranteed cost will increase when the policyholders have the right to rebalance his own account in every period. I can get the following conclusions. First, The guaranteed cost of variable annuities with GMWB when the policyholder can rebalance his own asset allocation in every year is about 12% more than the guaranteed cost of the fixed Iight which is calculated by the average of the Iights in each year of our optimal allocation. Second, the optimal allocation is affected by the asset model and the risk averse attitude of the policyholder in our model. When the asset model is a fat-tailed model, it needs more guaranteed cost; when the policyholder is more aggressive, the more guaranteed cost the insurance companies need to afford .