Actuarial risk classification: minimum bias
(Mathematical statistics modules 23 and 24)

Class relativities determined one dimension at a time are suitable for a single-dimensional class systems. Minimum bias and generalized linear models are needed when the class system has multiple dimensions. The examples here use two dimensions; the extension to more dimensions is straightforward, but the arithmetic and display are cumbersome.

We define the minimum bias terms, explain the statistical procedures, and review the intuition underlying each method. The basic illustration here has two dimensions with two values in each dimension. This prevents the intuition from getting submerged under tedious mathematics. In practice, the minimum bias procedure is most useful for multi-dimensional class systems with many entries in each dimension.

We show the computations for one iteration in each illustration and the series of values until convergence. The illustrations here converge in a few steps; more iterations are needed for larger models. Spreadsheets have built-in iterative functions, such as *goal-seek* and *solver* in Excel. Some software packages, such as SAS and R, have built-in routines for generalized linear models.

Generalized linear models for actuarial pricing often use Poisson distributions for accident frequency and log-link functions for multiplicative models, which is equivalent to the minimum bias balance principle with a multiplicative model. The balance principle is easier to understand and it is more intuitive: balancing the observed and expected values is a logical way to set rates.

MULTIPLICATIVE MODELS ASSUME PREMIUMS ARE THE PRODUCT OF CLASS RELATIVITIES.

This reading deals with pure premiums, not expenses, gross premiums, or profit. The pure premiums depend on the empirical observations in each cell of an array. For two dimensions, this means each cell in a matrix. The observations can be average loss costs, loss frequencies, or loss ratios. In practice, the data consist of losses and exposures (for loss costs), claim counts and exposures (for loss frequencies), or losses and premiums (for loss ratios).

Illustration 23.1: A motor insurance class system has dimensions of (i) urban vs rural and (ii) male vs female. A company insures four drivers, one in each cell, with the following observed loss costs:[1]

	Urban	Rural
Male	\$600	\$200
Female	\$300	\$100

We determine pure premium relativities. Comparing all males (\$800 for two exposures) with all females (\$400 for two exposures) gives a pure premium relativity of male to female = 2 to 1. Similarly, the relativity for urban vs rural is 3 to 1. Using "rural female" as the base class gives the following relativities:

Male:	2.00	=	s_1	Urban:	3.00	=	t_1
Female:	1.00	=	s_2	Rural:	1.00	=	t_2

The indicated pure premium for a male urban driver is the base pure premium times the urban relativity times the male relativity: \$100 × 2.00 × 3.00 = \$600. More generally, the pure premium in cell (i,j) is \$100 × s_i × t_j.

In this illustration, the indicated pure premiums exactly match the observed loss costs. The minimum bias method is not needed for this case.

ADDITIVE MODELS ASSUME PREMIUMS ARE THE BASE RATE TIMES THE SUM OF CLASS RELATIVITIES.

The indicated pure premiums may differ from observed loss costs because the model structure is incorrect or because random loss fluctuation distort the observed loss costs. We consider the model structure here.

Illustration 23.2: The observed loss costs for four drivers are shown below.

	Urban	*Rural*
Male	$700	$500
Female	$400	$200

We begin in the same fashion as before, using rural females as the base class. We compare all males to all females, giving a pure premium relativity of $1,200 to $600, or 2 to 1. We compare all urban to all rural, giving a pure premium relativity of $1,100 to $700, or 1.571 to 1.

The indicated pure premium relativities no longer match the observed loss costs. The indicated pure premium for rural males is $200 × 2.000 = $400, but the observed loss cost is $500. The indicated pure premium for urban females is $200 × 1.571 = $314, but the observed loss cost is $400.

No multiplicative factors are exact. In urban territories, the relation of male to female is $700 to $400, or 1.75 to 1. In rural territories, the relation of male to female is $500 to $200, or 2.50 to 1. A male-to-female relativity appropriate for urban territories is not appropriate for rural territories. Similarly, the urban to rural relativity is $700 to $500, or 1.4 to 1, for male drivers, and $400 to $200, or 2 to 1, for female drivers.

The discussion above assumes a multiplicative model; in this illustration, an additive model works better. We add or subtract a dollar amount for each class instead of multiplying by a factor. We choose rural females as the base class, and we use the relativities below.

- Male: +$300 Urban: +$200
- Female: +$0 Rural: +$0

The pure premium for any cell is the base pure premium plus the male/female relativity plus the urban/rural relativity. The indicated pure premiums now match the observed loss costs.

- urban male = $200 + $300 + $200 = $700
- rural male = $200 + $300 + $0 = $500
- urban female = $200 + $0 + $200 = $400
- rural female = $200 + $0 + $0 = $200

The additive method provides an exact match to the observed loss costs because the dollar differences are the same in each row ($200) and in each column ($300).

ADDITIVE AND MULTIPLICATIVE INTUITION ASSUMES ORTHOGONAL CLASS DIMENSIONS.

Some actuaries implicitly assume that pure premium relativities should be multiplicative, not additive. If urban male drivers have twice the accident frequency that rural male drivers have, urban female drivers should have twice the accident frequency that rural female drivers have. This assumption is most persuasive when class dimensions are independent: that is, when the high accident frequency of urban drivers is not correlated with the high accident frequency of male drivers.[2] Most multi-dimensional class systems for the casualty lines of business use multiplicative factors.

Some regulators castigate insurers for using multiplicative factors that "unduly" increase the rates for high-risk insureds. This criticism is often – but not always – political. When two or more dimensions of the class system are correlated, multiplicative systems are often biased. For some coverages, multiplicative systems may be biased even when class dimensions are not correlated.[3]

Class group and merit rating (bonus/malus systems) are not orthogonal. Class group is driver attributes, such as age, sex, credit score, and marital status, and use of the vehicle, such as pleasure use or business use. Merit rating class is the number of immediately preceding accident-free years, ranging from 0 to 3 or more.

These two rating dimensions are correlated. For example, young, unmarried male drivers have a high average class relativity. Because these drivers either are new drivers or (if not new) are more likely to have had an accident in the past year, they have relatively few accident free years, and a multiplicative model would penalize many young male drivers twice for the same risk.

Bias functions measure the fit between indicated pure premiums and observed loss costs.

Indicated pure premiums rarely match observed loss costs for either an additive or a multiplicative model. We illustrate with the observed loss costs in the table below:

	Urban	*Rural*
Male	$800	$500
Female	$400	$200

Neither an additive nor a multiplicative model is a perfect match. If the model does not match the observed data, we minimize the mismatch between the observed loss costs and the indicated pure premiums. A *bias function* selects relativities to fit the data with the smallest mismatch.[4] To choose the optimal model, we proceed along three steps:

1. We choose a rating method, such as an additive or a multiplicative model.
2. We select a bias function to optimize the rating. We examine balance principle, least squares, χ-squared, and maximum likelihood bias functions. For models using a maximum likelihood bias function, we must also choose a probability density function for losses within each cell.
3. For each optimized rating method, we examine the goodness-of-fit of the indicated pure premiums to the observed loss costs.

We begin with the balance principle, since it is the bias function most commonly used and it is equivalent to a Poisson GLM with a log-link function, the most common GLM used for actuarial pricing.

THE BALANCE PRINCIPLE SETS INDICATED PURE PREMIUMS TO OBSERVED LOSS COSTS ALONG ROWS AND COLUMNS.

The balance principle requires that (after optimizing the relativities) the sum of the *indicated pure premiums* equals the sum of the *observed loss costs* along every row and every column.

Illustration 23.3: We examine the balance principle for both additive and multiplicative models. There is one exposure in each cell. On the left are the observed loss costs; on the right are the indicated pure premiums. We begin with the multiplicative model.[5]

	Urban	*Rural*		*terr*$_1$	*terr*$_2$
Male	\$800	\$500	*sex*$_1$	$200 \times s_1 \times t_1$	$200 \times s_1 \times t_2$
Female	\$400	\$200	*sex*$_2$	$200 \times s_2 \times t_1$	$200 \times s_2 \times t_2$

- Balancing along the first row (males) gives $800 + 500 = 200 \times s_1 \times t_1 + 200 \times s_1 \times t_2$
- Balancing along the second row (females) gives $400 + 200 = 200 \times s_2 \times t_1 + 200 \times s_2 \times t_2$
- Balancing along the first column (urban) gives $800 + 400 = 200 \times s_1 \times t_1 + 200 \times s_2 \times t_1$
- Balancing along the second column (rural) gives $500 + 200 = 200 \times s_1 \times t_2 + 200 \times s_2 \times t_2$

We have four equations in four unknowns, but we do not have a unique solution for the class relativities. Two considerations offset each other to yield a unique set of indicated pure premiums for each cell of the matrix.

Dependence among the equations: The equations are related by a totality constraint – any three equations imply the fourth, since the sum of the rows equals the sum of the columns. For instance, the fourth equation equals the first equation plus the second equation minus the third equation. For a matrix with many rows and columns, the equation for any column is the sum of the row equations minus the sum of the equations for the other columns, and likewise for the equation in any row.

Invariance under reciprocal multiplication: We can set any one variable arbitrarily, and still solve the system of equations. If we solve the equations for values of the variables s_1, s_2, t_1, and t_2, another solution is $2s_1$, $2s_2$, $½t_1$, and $½t_2$. No matter which set of relativities we pick, the values in the cells do not change. These values are the product of an *s* relativity and a *t* relativity; the additional constant cancels out.

We have an additional variable. The pure premium in each cell depends on the base pure premium. If the relativities s_1, s_2, t_1, and t_2 optimize the rating model for a base pure premium of \$200, the relativities $2s_1$, $2s_2$, t_1, and t_2 optimize the rating model for a base pure premium of \$100.[6] The relativities within each class stay the same. If $s_1 = 2 \times s_2$ for a given base pure premium and a given set of territorial relativities, then $s_1 = 2 \times s_2$ for any other base pure premium and for any constant multiple of the territorial relativities.

We choose a base class for each class dimension, such as the class with the most insureds or the lowest-cost class (though any class may be used). The base class in each class dimension has a relativity of 1. The base classes determine the values of the base pure premium and the other rating variables.

The equations are not linear, but they are easily solved iteratively.

The equations are not linear, so there is no closed-form solution. We begin with an arbitrary (but reasonable) set of relativities for one dimension, and we solve the equations iteratively.

Illustration 23.4: We choose an urban relativity of 2.00 and rural relativity of 1.00; this initial choice does not affect the final pure premiums.

	Urban	*Rural*		*terr*$_1$ *= 2*	*terr*$_2$ *= 1*
Male	\$800	\$500	*sex*$_1$	$200 \times s_1 \times 2$	$200 \times s_1 \times 1$
Female	\$400	\$200	*sex*$_2$	$200 \times s_2 \times 2$	$200 \times s_2 \times 1$

We balance (equate) the observed values with the theoretical values.

- Balancing along the male row gives $800 + 500 = 200 \times s_1 \times 2 + 200 \times s_1 \times 1 \Rightarrow s_1 = 1{,}300/600 = 13/6$.
- Balancing along the female row gives $400 + 200 = 200 \times s_2 \times 2 + 200 \times s_2 \times 1 \Rightarrow s_2 = 600/600 = 1$.

We now have intermediate values for the male and female relativities of 13/6 and 1. We discard the initial values for the urban and rural relativities of 2.00 and 1.00 and solve for new intermediate values by balancing down the columns:

- Balancing down the urban column: 800+400 = 200×(13/6) × t_1 + 200 × 1 × t_1 ⇛ t_1 = 1,200/633.33 = 1.895.
- Balancing down the rural column:500 + 200 = 200 × (13/6) × t_2 + 200 × 1 × t_2 ⇛ t_2 = 1.105.

We continue in this fashion. We discard the previous male and female relativities, and we solve for new ones. Balancing along the male row gives 800 + 500 = 200 × s_1 × 1.895 + 200 × s_1 × 1.105 and balancing along the female row gives 400 + 200 = 200 × s_2 × 1.895 + 200 × s_2 × 1.105. We solve these two equations for new values of the male and female relativities, we discard the previous values of the urban and rural relativities, and we balance along the columns for new values of the urban and rural relativities.

We continue in this fashion until the relativities converge, i.e., the change in the relativities from an additional iteration is not material. Calculating minimum bias relativities is tedious by hand but easy with a spreadsheet. In this case, convergence is rapid, since there are only four cells. Once the series converges, we normalize the base class relativities to unity and change the base rate (to $221.05), as we do below.

Iteration	Urban	Rural	Male	Female
Initial	2.0000	1.0000		
1-a			2.1667	1.0000
1-b	1.8947	1.1053		
2-a			2.1667	1.0000
Final	1.8947	1.1053		
Normalized	1.7143	1.0000	2.1667	1.0000
Normalized Base Pure Premium:			$200 × 1.1053 = $221.05	

The initial territorial relativities of 2.00 and 1.00 were arbitrary; we might begin with starting values determined by a *one way* relativities procedure. The starting values have no effect on the final rates in each cell, though better starting values reduce the iterations required to reach convergence. In this illustration, the urban to rural relativity is 12 to 7. If we choose 1.000 as the starting value for the rural class, we would choose 12 ÷ 7 = 1.714 for the urban class. With a starting value of t_1 = 1.714, the series converges immediately; for a two-by-two matrix, the one-way relativities are the solution. We used a different starting value to show the procedure.

MULTIPLE OBSERVATIONS IN EACH CELL

We have assumed so far a single observation in each cell (or the same number of observations in each cell). If the number of observations differs by cell, we treat each observation as a separate cell. In practice, pricing actuaries have the total value (or mean value) for each cell and the number of observations by cell (such as the total losses by cell and the number of claims by cell), so we weight the cell by the number of observations in the cell.

To illustrate, suppose the cells have 19, 17, 13, and 11 observations, as shown below:

Mean value	*Urban*	*Rural*	*Observations*	*$terr_1$ = 2*	*$terr_2$ = 1*
Male	$800	$500	*sex_1*	19	17
Female	$400	$200	*sex_2*	13	11

Balancing along the rows or down the columns gives

- first row (males) $19 \times 800 + 17 \times 500 = 19 \times 200 \times s_1 \times t_1 + 17 \times 200 \times s_1 \times t_2$
- second row (females) $13 \times 400 + 11 \times 200 = 13 \times 200 \times s_2 \times t_1 + 11 \times 200 \times s_2 \times t_2$
- first column (urban) $19 \times 800 + 13 \times 400 = 19 \times 200 \times s_1 \times t_1 + 11 \times 200 \times s_2 \times t_1$
- second column (rural) $17 \times 500 + 11 \times 200 = 17 \times 200 \times s_1 \times t_2 + 11 \times 200 \times s_2 \times t_2$

The computed male and female relativities are

- Balancing along the male row gives
  - $19 \times 800 + 17 \times 500 = 19 \times 200 \times s_1 \times 2 + 17 \times 200 \times s_1 \times 1 \Rrightarrow$
  - $s_1 = (19 \times 800 + 17 \times 500) / (19 \times 200 \times 2 + 17 \times 200 \times 1) = 2.154545$
- Balancing along the female row gives
  - $13 \times 400 + 11 \times 200 = 13 \times 200 \times s_2 \times 2 + 11 \times 200 \times s_2 \times 1 \Rrightarrow$
  - $s_2 = (13 \times 400 + 11 \times 200) / (13 \times 200 \times 2 + 11 \times 200 \times 1) = 1.00$

We now discard the initial values for urban and rural relativities of 2.00 and 1.00 and solve for the relativities by balancing down the columns.

- Balancing along the urban column:
  - $19 \times 800 + 13 \times 400 = 19 \times 200 \times 2.154545 \times t_1 + 13 \times 200 \times 1 \times t_1 \Rrightarrow$
  - $t_1 = (19 \times 800 + 13 \times 400) / (19 \times 200 \times 2.154545 + 13 \times 200 \times 1) = 1.891118$
- Balancing along the rural column:
  - $17 \times 500 + 11 \times 200 = 17 \times 200 \times 2.154545 \times t_2 + 11 \times 200 \times 1 \times t_2 \Rrightarrow$
  - $t_2 = (17 \times 500 + 11 \times 200) / (17 \times 200 \times 2.154545 + 11 \times 200 \times 1) = 1.123306$

ADDITIVE MODELS HAVE SEVERAL FORMS.

Additive models have several equivalent forms. The pure premium in cell (i,j), or row "i" and column "j," is

A. Base pure premium + $x_i + y_j$,
B. Base pure premium × $(1 + u_i + v_j)$, or
C. Base pure premium × $(p_i + q_j)$

To see the equivalence of these formulas, suppose the base pure premium in formula "A" is $10.

- In formula "B," the base pure premium is also $10, each "u" value is one tenth the corresponding "x" value in formula "A," and each "v" value in formula "B" is one tenth the corresponding "y" value in formula "A": $u_i = 0.1 \times x_i$ and $v_j = 0.1 \times y_j$.
- Formula "C" is equivalent to formula "B," except that either the "p" values are all increased by 1, the "q" values are all increased by 1, or the "p" values are increased by a constant (*c*) and the "q" values are increased by the complement of that constant (*1–c*): $p_i = 1 + u_i$ <u>*or*</u> $q_j = 1 + v_j$ (but not both) or p_i = "c" + u_i <u>*and*</u> q_j = "1–c" + v_j.

We use formula *A* for our example, since it shows the method most simply.[7]

Illustration 23.5: We choose initial urban and rural relativities of $250 and $0, based on one-way relativities: the average difference between urban and rural loss costs is ½ × [(800 – 500) + (400 – 200)] = $250.

	Urban	*Rural*		*$terr_1$ = 250*	*$terr_2$ = 0*
Male	$800	$500	*sex_1*	$200 + s_1 + 250$	$200 + s_1 + 0$
Female	$400	$200	*sex_2*	$200 + s_2 + 250$	$200 + s_2 + 0$

- Balancing along the male row gives $800 + 500 = 200 + s_1 + 250 + 200 + s_1 + 0 \Rightarrow s_1 = 650/2 = 325$.
- Balancing along the female row gives $400 + 200 = 200 + s_2 + 250 + 200 + s_2 + 0 \Rightarrow s_2 = -50/2 = -25$.

We discard the initial values for the urban and rural relativities, and we balance along the columns. We use the intermediate values of the male and female relativities to get new values for the urban and rural relativities. We continue this iterative process until the series converges.

A negative relativity of –$25 for females might seem odd. In truth, the relativity is not inherently negative; this is an artifact of the base pure premium and the starting values. We could make the relativity for females positive by adding a constant to the male and female relativities and subtracting the same constant from the rural and urban relativities. For instance, we could add $75 to the male and female relativities to get relativities of $400 and $50, and we would subtract $75 from the rural and urban relativities.

Iteration	Urban	Rural	Male	Female
Initial	$250.00	$0.00		
1-a			$325.00	($25.00)
1-b	$250.00	$0.00		
Normalized	$250.00	$0.00	$350.00	$0.00
Normalized Base Pure Premium:			$200.00 – $25.00 = $175.00	

We can even make all the relativities negative by adjusting the base pure premium. For instance, by choosing a base pure premium of $1,000, we obtain negative relativities for all classes.[8] In this illustration, we added dollar amounts to make the base class relativities equal to zero.

EXPOSURES ARE ADDITIONAL CELLS; THE MINIMUM BIAS PROCEDURE DOES NOT CHANGE.

The illustrations above assume one driver (or the same number of drivers) in each cell. With different numbers of risks in each cell, two adjustments are needed: one to the bias function and another for credibility.

- We adjust the bias function for the relative volume of business in each cell.
- We may make a credibility adjustment based on the absolute volume of business in a cell.

For credibility, suppose insurer A has 100 exposures per cell, and insurer B has 10,000 exposures per cell. Insurer A may rely more on the minimum bias procedure; Insurer B may give more weight to the observations.

The balance principle requires the sum of the observed loss costs in a row or column to equal the sum of the indicated pure premiums in the same row or column. Two drivers in a cell doubles both the observed loss cost and the indicated pure premium in that cell. For *n* drivers in a cell, we multiply the observed loss cost and the indicated pure premium by *n*. If the number of drivers varies by cell, we use a matrix of the number of drivers in each cell.

Illustration 23.6: For the multiplicative model, suppose that the number of drivers is as follows:

- Male urban: 1,200 Male rural: 600
- Female urban: 1,000 Female rural: 800

We include the number of drivers in the equations.

	Urban	Rural		$terr_1$	$terr_2$
Male	1200 × \$800	600 × \$500	sex_1	$1200 \times 200 \times s_1 \times t_1$	$600 \times 200 \times s_1 \times t_2$
Female	1000 × \$400	800 × \$200	sex_2	$1000 \times 200 \times s_2 \times t_1$	$800 \times 200 \times s_2 \times t_2$

Balancing along the male row gives: $1200 \times 800 + 600 \times 500 = 1200 \times 200 \times s_1 \times t_1 + 600 \times 200 \times s_1 \times t_2$
Balancing along the female row gives: $1000 \times 400 + 800 \times 200 = 1000 \times 200 \times s_2 \times t_1 + 800 \times 200 \times s_2 \times t_2$
Balancing down the urban column: $1200 \times 800 + 1000 \times 400 = 1200 \times 200 \times s_1 \times t_1 + 1000 \times 200 \times s_2 \times t_1$
Balancing down the rural column gives: $600 \times 500 + 800 \times 200 = 600 \times 200 \times s_1 \times t_2 + 800 \times 200 \times s_2 \times t_2$

Empirical Observations vs Modeled Pure Premiums

One might wonder: Why not use observed loss costs, appropriately developed and trended, as the indicated pure premiums for the coming policy period? Instead of fitting either multiplicative or additive models to the observed data, use \$800 as the indicated pure premium for urban male drivers, \$400 for urban female drivers, \$500 for rural male drivers, and \$200 for rural female drivers.

The common answer is that the individual cells are *not fully credible*. This answer is correct, though the term *credible* is vague. To grasp the intuition, we must be more precise.

Credibility is a relative concept. No cell is inherently credible or not credible. A cell's credibility depends on the reliability of its own experience in comparison with information in other cells. Consider the illustration with the following observed loss costs:

	Urban	Rural
Male	\$800	\$500
Female	\$400	\$200

The urban male observed pure premium of \$800 represents a mixture of expected losses and random loss fluctuations. How might we judge whether this figure is higher or lower than the true expected loss costs?

Suppose that the rating values combine additively to generate the expected losses. The observed loss cost for urban males of \$800 is \$300 more than the observed loss cost of rural males of \$500, implying that the urban attribute of the vehicle's location adds \$300 to the expected loss costs. But the urban female observed loss cost of \$400 is only \$200 more than the rural female observed loss cost of \$200, suggesting that

- the extra cost associated with the urban attribute is only \$200
- the observed urban male loss cost of \$800 might be too high.

We perform a similar analysis for male versus female. Comparing urban drivers suggests that the male attribute adds about \$400 to the expected loss costs, since male-urban = \$800 and female-urban = \$400. However, comparing rural drivers suggests that

- the extra cost associated with being male is \$300, since male-rural = \$500 and female-rural = \$200
- the urban male loss cost of \$800 might be too high.

The \$800 in the urban male cell does not tell us what part of this observed loss cost is expected and what part is random loss fluctuations. If we know the mathematical function linking the cells – that is, whether the pure premiums have an additive or multiplicative relation – we can use additional cells to provide information about the true expected costs for urban male drivers, as we have here.

If the cells are linked multiplicatively, our inferences change. The urban male observed value of $800 is 160% of the rural male observed value of $500, suggesting that the urban attribute adds 60% to the expected losses. The urban female observed loss cost of $400 is twice the rural female observed loss cost of $200, suggesting that the extra cost from the urban attribute is +100%, so the urban male loss cost of $800 might be too low.

A similar analysis for male versus female using the urban column suggests that the male attribute adds about 100% to the expected loss costs. The rural column suggests that the extra cost of the male attribute is +150%, so the urban male loss cost of $800 might be too low.

If the cells are linked additively, we infer that the urban male observed loss costs of $800 might be too high. If they are linked multiplicatively, we infer that the urban male observed loss costs of $800 might be too low.[9]

If the exposures in a 2 by 2 matrix are evenly distributed among the cells, each cell has 25% of the total exposures, whether there is 1 car or 10,000 cars in each cell. We give much credence to the observed value in that cell compared to our inferences from other cells. With a larger array, such as a 10 by 10 by 10 array, there are many more cells. The average cell has only 0.1% of the total exposures, so we give less credence to the observed loss costs in that cell compared to our inferences from other cells.

We summarize the intuition for the minimum bias pricing procedure in two statements:

- The rating model – such as additive, multiplicative, or combined – tells us the relation joining the cells.
- The bias function – such as balance principle, χ-squared, least squared error, or maximum likelihood – provides a method of drawing inferences for one cell using the information in the other cells.

ITERATIVE FORMULAS SIMPLIFY SPREADSHEET COMPUTATIONS.

We have so far presented simple illustrations. To program the procedure, we need general formulas.

We derive the iterative formulas for the multiplicative balance principle model. We designate the base pure premium by *b*, the number of exposures in row *i* and column *j* by n_{ij}, and the observed pure premiums in row *i* and column *j* by r_{ij}. With a multiplicative model, the balancing equation for row *i* is

$$\sum_j (n_{ij} r_{ij}) = \sum_j (b n_{ij} x_i y_j)$$

Similarly, the balancing equation for column *j* is

$$\sum_i (n_{ij} r_{ij}) = \sum_i (b n_{ij} x_i y_j)$$

In these equations, *x* is a row relativity and *y* is a column relativity.[10] We solve these equations to get the indicated *x* and *y* relativities in each row and column:[11]

$$x_i = \frac{\sum_j n_{ij} r_{ij}}{\sum_j b n_{ij} y_j} \text{ and } y_j = \frac{\sum_i n_{ij} r_{ij}}{\sum_i b n_{ij} x_i}$$

When the series converges, we set the relativity for the base class in each class dimension to unity, and we adjust the base pure premium to offset this.

This formula has two dimensions, and one might infer that the two dimensions correspond to the two variables *x* and *y*. That is not correct; the two dimensions correspond to the subscripts *i* and *j*. The *x* and *y* variables are two sets of relativities. A model can have two or even more sets of relativities in a single dimension.

Illustration: A class system has two dimensions: male vs female and territory A vs territory B. Territory A has more attorneys than territory B has, resulting in a higher propensity to sue and higher loss costs per claim. Territory B has several blind intersections, leading to more accidents. We presume that the higher attorney involvement in territory A increases the cost of all claims, so a multiplicative factor is appropriate, and the blind intersections in territory B adds additional hazards, so an additive factor is appropriate. The rating model might take the form

indicated pure premium relativity = $x_i \times y_j + z_j$,

where *i* represents the male/female class dimension and *j* represents the territory dimension. The variable *x* is the relativity for the male/female dimension, and the variables *y* and *z* are the relativities for the urban/rural dimension. In this model, *x* and *y* are multiplicative factors, and *z* is an additive factor.[12]

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The arithmetic is similar for any number of dimensions. The multiplicative model has one set of relativities for each dimension. With three dimensions, for example, the iterative formula for the i^{th} dimension is

$$x_i = \frac{\sum_{j,k} n_{ijk} r_{ijk}}{\sum_{j,k} b n_{ijk} y_j z_k}$$

We develop the general formula for the balance principle additive model by assuming a base pure premium of $0. The balance principle equation is

$$\sum_j (n_{ij} r_{ij}) = \sum_j n_{ij}(x_i + y_j)$$

and the iterative formula is

$$x_i = \frac{\sum_j n_{ij}(r_{ij} - y_j)}{\sum_j n_{ij}}$$

EXERCISE: MULTIPLICATIVE MODEL WITH EXPOSURES

Exercise 33.1: We use the balance principle to optimize a multiplicative rating model with two dimensions and two classes in each dimension. The observed loss costs and exposures in each class are

Loss Costs	y_1	y_2	Exposures	y_1	y_2
x_1	300	300	x_1	100	150
x_2	200	400	x_2	100	100

We assume a base pure premium of \$100, so the indicated pure premiums are \$100x_iy_j. To simplify the mathematics, we compute all values in units of \$100. The indicated pure premiums are $x_i \times y_j$, and the observed loss costs are \$3, \$3, \$2, and \$4.

We form a matrix of observed loss costs and indicated pure premiums:

	y_1	y_2		y_1	y_2
x_1	3	3	x_1	$x_1 \times y_1$	$x_1 \times y_2$
x_2	2	4	x_2	$x_2 \times y_1$	$x_2 \times y_2$

We multiply each figure by the exposures in the cell:

	y_1	y_2		y_1	y_2
x_1	100 × 3	150 × 3	x_1	$100 \times x_1 \times y_1$	$150 \times x_1 \times y_2$
x_2	100 × 2	100 × 4	x_2	$100 \times x_2 \times y_1$	$100 \times x_2 \times y_2$

We choose 1.0 and 1.5 as starting values for y_1 and y_2, and use the balance principle to obtain intermediate values for x_1 and x_2:

$$100 \times 3 + 150 \times 3 = 100 \times x_1 \times 1.00 + 150 \times x_1 \times 1.50,$$
$$\text{or } 300 + 450 = 100 \times x_1 + 225 \times x_1,$$
$$\text{or } x_1 = 2.308.$$

and

$$100 \times 2 + 100 \times 4 = 100 \times x_2 \times 1.00 + 100 \times x_2 \times 1.50,$$
$$\text{or } 200 + 400 = 100 \times x_2 + 150 \times x_2,$$
$$\text{or } x_2 = 2.400.$$

We now discard the initial values for y_1 and y_2, and we balance along the columns.

$$100 \times 3 + 100 \times 2 = 100 \times 2.308 \times y_1 + 100 \times 2.400 \times y_1,$$
$$\text{or } 300 + 200 = 230.8 \times y_1 + 240 \times y_1,$$
$$\text{or } y_1 = 1.062.$$

and

$$150 \times 3 + 100 \times 4 = 150 \times 2.308 \times y_2 + 100 \times 2.400 \times y_2,$$
$$\text{or } 450 + 400 = 346.2 \times y_2 + 240 \times y_2,$$
$$\text{or } y_2 = 1.450.$$

This completes one iteration. To solve for the optimal relativities, we continue in this fashion until convergence.

Comments on the data and assumptions in the exercise:

The number of exposures in each cell is a credibility measure. We give 50% more credence to the observed loss costs in the x_1y_2 cell than to the loss costs in the other cells.

- The observed loss costs in the x_1 row indicate that there is no difference between y_1 and y_2. The observed loss costs in the x_2 row indicate that the y_2 class should have twice the pure premium that y_1 has. We give more credence to the first of these two relations.
- The observed loss costs in the y_1 column indicates that the x_2 class should have a pure premium 33% lower than the x_1 class. The observed loss costs in the y_2 column indicates that the x_2 class should have a pure premium 33% higher than the x_1 class. We give more credence to the second of these relations, so the x_2 relativity is slightly higher than the x_1 relativity.

EXERCISE: ADDITIVE MODEL

Exercise 33.2: An additive model with two dimensions has the observed loss costs shown below. Each cell has 1,000 exposures. The base lost cost is 100. The formula for loss costs by cell is Loss $Cost_{ij}$ = (Base Loss Cost) × $(x_i + y_i)$. We use the starting values below to compute intermediate values for y_1 and y_2.

Average Loss Costs / Exposure:	y_1	y_2	Starting Values:	
x_1	500	750	x_1	4.500
x_2	250	475	x_2	3.000
x_3	150	400	x_3	2.000

Since the exposures are the same in each cell, the *1,000* cancels out of all equations. The base pure premium is \$100. To simplify, we use units of \$100 and a base pure premium of \$1. The matrix of observed loss costs and indicated pure premiums is shown below:

	Observed Values			*Indicated Values*	
	y_1	y_2		y_1	y_2
x_1	5	7.5	x_1	$x_1 + y_1$	$x_1 + y_2$
x_2	2.5	4.75	x_2	$x_2 + y_1$	$x_2 + y_2$
x_3	1.5	4	x_3	$x_3 + y_1$	$x_3 + y_2$

Balancing down the first column gives $5.00 + 2.50 + 1.50 = (x_1 + y_1) + (x_2 + y_1) + (x_3 + y_1)$

Substituting the starting values of the “x”s gives

$$5.00 + 2.50 + 1.50 = (4.50 + y_1) + (3.00 + y_1) + (2.00 + y_1),$$
$$\text{or } 3\,y_1 = 9.00 – 9.50, \text{ or } y_1 = –0.167.$$

For the second column:

$$7.50 + 4.75 + 4.00 = (x_1 + y_2) + (x_2 + y_2) + (x_3 + y_2)$$
$$7.50 + 4.75 + 4.00 = (4.50 + y_2) + (3.00 + y_2) + (2.00 + y_2),$$
$$\text{or } 3\,y_2 = 16.25 – 9.50, \text{ or } y_2 = 2.25.$$

We next balance along the rows: using the values y_1 = –0.167 and y_2 = +2.25, we compute new values for x_1 and x_2 We alternately balance along rows and columns until the figures converge. During the iterative process, the plan is alternately balanced along the rows or along the columns, but not along both. We have just balanced along the columns. To see that we are not yet balanced along the rows, we examine the first row:

$$5.00 + 7.50 = (x_1 + y_1) + (x_1 + y_2).$$

Substituting the starting values of the "x"s and the first iterative values of the "y"s, we get

$$12.50 = 4.50 + (-0.167) + 4.50 + 2.25 = 11.083.$$

The equality does not hold, since the plan is not yet balanced. The final values, after convergence, are

Iteration	x_1	x_2	x_3	y_1	y_2
Initial	4.50000	3.00000	2.00000		
1-a				-0.16667	2.25000
1-b	5.20833	2.58333	1.70833		
2-a				-0.16667	2.25000
Final	5.20833	2.58333	1.70833	-0.16667	2.25000

Squared Error vs χ-Squared

The squared error bias function is similar to the χ-squared bias function, but whereas the squared error test looks at absolute differences, the χ-squared test looks at percentage differences. Some statisticians prefer the χ-squared test to a least squares test.

Illustration 33.7: We are fitting a distribution to two empirical data points.

- Point A has an observed value of \$101 and a fitted value of \$100.
- Point B has an observed value of \$1.50 and a fitted value of \$1.00.

We examine the errors for each point.

- The squared error is $(101 - 100)^2 = 1.00$ for point A and $(1.50 - 1.00)^2 = 0.25$ for point B. This distribution fits point B better.
- The χ-squared value is $(101 - 100)^2 / 100 = 0.01$ for point A and $(1.50 - 1.00)^2 / 1.00 = 0.25$ for point B. This distribution fits point A better.

The statistician might prefer the χ-squared test to the squared error test. The practical businessperson might argue that the insurance enterprise is not concerned with optimizing a statistical fit. It is concerned with optimizing net income. At point A, the insurer has a gain or loss of \$1.00. At point B, the gain or loss is \$0.50. The squared error test is preferred.

This argument would be correct if we fully believed the observed values – that is, if the observed values were fully credible. But if the observed values were fully credible, we would have no need to use the minimum bias procedure; we would just use the rates indicated by the observed loss costs in each cell.

We use the minimum bias procedure because the observed values are not fully credible, and the relations among the cells in the observed matrix provides information for choosing the true expected values. When we say that a particular fit "X" has less of an error than another fit "Y," we do not mean that we know the true values and that model "X" is closer to these true values. We do not know the true values, but we presume that these true values might be represented by a mathematical function. When we say that fit “X” is better, we mean that model “X” is more likely to be a better model. The χ-squared bias function perhaps does a better job of choosing the better model. If so, the businessperson might also prefer the χ-squared bias function.

BALANCE PRINCIPLE VS X-SQUARED

The 1960 Bailey and Simon paper prefers the χ-squared bias function to the balance principle, whereas the 1963 Bailey paper argues for the balance principle. In defense of the χ-squared bias function, the 1960 Bailey and Simon paper says (page 10):

> *The indication of each group should be given a weight inversely proportional to the standard deviation of the indication.*

This is a traditional justification for classical credibility, as Bailey and Simon continue:

> *The standard deviation of the indication is inversely proportional to the square root of the expected number of losses for the group.*[13]

The 1963 Bailey paper prefers the balance principle because it is unbiased, whereas the χ-squared bias function may be biased. An unbiased function constrains the relativities so that the total indicated pure premiums along any dimension equal the total observed loss costs along that dimension. By definition, the balance principle is unbiased, whereas other functions may be biased.

The balance principle uses the first-order departure, which is generally preferred by firms seeking to maximize profits. This is perhaps the strongest argument for the balance principle.

Most actuaries use the balance principle. If more effective methods drive out less effective methods in a competitive market, the balance principle is perhaps the most effective bias function.

In truth, many ratemaking procedures are selected for ease of implementation, not necessarily for accuracy. The balance principle was easier to implement before the widespread use of desktop computers. Few actuaries have tried the χ-squared bias function or the least squares bias function.

Other Class Dimensions

The illustrations set pure premium relativities for the male/female dimension and the urban/rural dimension. There may be other dimensions to the class plan as well, such as age of driver, marital status, type of vehicle, use of the car, driver education, and prior accident history.

Suppose that we analyze the male/female dimension and the urban/rural dimension on a statewide basis, and we set relativities for other class dimensions on a countrywide basis. We use a minimum bias method for the statewide analysis.

If all the class dimensions are independent, the analysis works well. If other class dimensions are correlated with the male/female or urban/rural dimension, the analysis may be distorted.

Illustration 33.8: Suppose young people live in urban areas, for university education, work opportunities, and urban social activities. Older people live in suburbs and rural areas, to buy homes and raise families away from urban areas. The age and marital status of the driver are correlated with the urban/rural dimension.

The statewide analysis may indicate an urban to rural relativity of 2 to 1. The countrywide analysis, summing over all territories, may indicate a relativity for young unmarried male drivers of 3 to 1 when compared to adult drivers. The relativity for young unmarried urban male drivers is not 6 to 1, even if a multiplicative model is appropriate for automobile insurance, since many young unmarried male drivers in the countrywide analysis live in urban areas, and many urban drivers in the statewide analysis are young and unmarried.

MULTIPLE DIMENSIONS

Ideally, we use a minimum bias procedure to set all class relativities simultaneously. In practice, this may not be possible. Some relativities may be analyzed each year; other relativities may be analyzed every several years. Some relativities, such as territory, must be set on a statewide basis. Certain driver characteristics and vehicle characteristics may be analyzed on a countrywide basis, for two reasons:[14]

1. The relativities do not vary by state, as long as the states use the same insurance compensation system.
2. Some class cells have few exposures in a state analysis, and the results may be distorted by random loss fluctuations. The countrywide analysis uses more data, providing more credible results. For example, we may analyze driver age in yearly increments: age 17, age 18, age 19, and so forth. Single-state data may be too sparse to give credible results.

Some class dimensions, such as driver education, have a minor effect on overall loss costs. We may analyze these class dimensions every five years, not every year.

Loss Ratios

One method of dealing with an uneven distribution of business along other class dimensions is to use loss ratios instead of loss costs in the minimum bias procedure.[15] [16]

Suppose the empirical experience consists of the following loss ratios by class.

	Urban	*Rural*
Male	75%	85%
Female	90%	80%

We could take either of two approaches:

First Approach: We treat the unadjusted loss ratios as loss costs. Instead of using pure premium relativities, we develop loss ratio relativities. These relativities are adjustments to whatever pure premium relativities are embedded in these loss ratios.

In this scenario, the minimum bias procedure will indicate a loss ratio relativity close to 1.000 for urban vs rural and a relativity slightly higher than 1.000 for females versus males. This does not mean that urban risks are similar to rural risks, or that female drivers have more accidents than male drivers have. If the current rate relativities are reasonable, we would expect the loss ratios in all cells to be about equal. Suppose that the current male-to-female rate relativity is 2.4 to 1. Since the average female loss ratio of 85% is higher than the average male loss ratio of 80%, the loss ratio relativities would indicate that we should slightly reduce the male-to-female rate relativity.

Second Approach: We convert the observed loss ratios to base class loss ratios. Suppose the current rate relativities are 2.4 to 1 for male to female and 1.8 to 1 for urban to rural. We divide the male premiums by 2.4 and the urban premiums by 1.8. This is equivalent to multiplying the male loss ratios by 2.4 and the urban loss ratios by 1.8. We multiply the raw loss ratios by the current class relativities, as shown in the table below.

	Urban	*Rural*
Male	75% × 2.4 × 1.8 = 324%	85% × 2.4 × 1.0 = 204%

Female	90% × 1.0 × 1.8 = 162%	80% × 1.0 × 1.0 = 80%

We apply the minimum bias procedure to the adjusted loss ratios. The resulting loss ratio relativities would be the same as the indicated rate relativities. Suppose the base rate is $100.

- For the male/urban cell, the premium is $100 × 2.4 × 1.8 = $432. The observed loss ratio is 75%, so the loss cost is 75% × $432 = $324.
- For the male/rural cell, the premium is $100 × 2.4 × 1.0 = $240. The observed loss ratio is 85%, so the loss cost is 85% × $240 = $204.
- For the female/urban cell, the premium is $100 × 1.0 × 1.8 = $180. The observed loss ratio is 90%, so the loss cost is 90% × $180 = $162.
- For the female/rural cell, the premium is $100 × 1.0 × 1.0 = $100. The observed loss ratio is 80%, so the loss cost is 80% × $100 = $80.

To set the rate relativity to unity for the base class in each dimension, we divide each adjusted loss ratio in the matrix by the adjusted loss ratio for the base class.

	Urban	*Rural*
Male	324% / 80% = 405.0%	204% / 80% = 255.0%
Female	162% / 80% = 202.5%	80.0% / 80% = 100.0%

Loss Ratio Intuition

We have shown how to convert loss ratios to reflect the loss costs in each cell, assuming the observed data are loss ratios and we want to use loss costs for the minimum bias procedure. If observed data are loss costs, we must convert the observed loss costs to loss ratios before converting back to loss costs. The purpose of converting from loss costs to loss ratios and then back to loss costs is to eliminate distorting effects of other class dimensions that are not being analyzed.

Illustration 33.9: We explain by illustration. We have average observed bodily injury loss costs for four groups of drivers, with 1,000 drivers in each cell.

	Urban	*Rural*
Male	$800	$500
Female	$400	$200

Other dimensions in the class system are correlated with the two dimensions above.

Type of Vehicle: For bodily injury rating, cars are subdivided between (a) large cars, such as sports utility vehicles (SUV’s), station wagons, and light trucks, and (b) small cars, such as sedans. Large vehicles provide better protection for their occupants but cause greater damage to others. Smaller vehicles cause less damage to others but provide less protection for their occupants. Small cars are more common in urban areas; SUV’s and light trucks are more common in rural areas. The distribution of vehicle types between urban and rural areas, along with the surcharge or discount for each type of vehicle, affects the observed loss costs.

Suppose SUV’s and other large vehicles receive a 20% surcharge for bodily injury. In this state, SUV’s are 40% of the rural vehicles and 10% of the urban vehicles. The pricing actuary may not have this distribution; the use of loss ratios instead of loss costs corrects for the effects of vehicle type.

Age of Driver: The male/female rate relativity applies to all male and female drivers. Unmarried male drivers under the age of 21 receive additional surcharges, ranging from 25% for 20-year-old drivers to 125% for 16-year-old drivers. In this state, 10% of male drivers are unmarried and under the age of 21. The average surcharge for these drivers is 50%. The pricing actuary may not have a distribution of male drivers by age and marital status; the loss ratios are sufficient.

Double Counting and Offsetting

If we do not take vehicle type and driver age into account, we overcharge male drivers and rural drivers.

Male Drivers: The male/female relativity is based on the statewide analysis. The surcharge for young unmarried male drivers is determined from a separate countrywide analysis. The poor driving experience of young unmarried male drivers is counted twice: once at the countrywide level for the surcharges and once at the state level for the male/female relativity. To determine the male/female relativity, we must remove the hazardous effects of being young and unmarried from the male driver class.

Rural Drivers: Rural drivers are less hazardous than urban drivers, but they drive vehicles more dangerous to others. The vehicle surcharge is determined in a countrywide analysis. To determine the urban/rural relativity, we must remove the effects of vehicle type from the statewide experience.

To remove the effects of vehicle type and driver age from the statewide analysis, we assume that the countrywide relativities are accurate. We examine each risk in the minimum bias procedure. We divide the actual loss costs by the vehicle type relativity and by the driver age relativity. This gives the relative loss costs that we would have expected to see were the vehicle types and driver ages evenly distributed over all other rating dimensions.

Illustration 33.10: continued: A four-door sedan is the base vehicle type and age 21+ is the base age. A two-door compact has a bodily injury discount of 10%, and an unmarried 20-year-old male driver has a surcharge of 25%. If the observed loss costs for a 20-year-old unmarried male driver of a two door compact car are $450, the loss costs adjusted for driver age and vehicle type are *$450 / (0.90 × 1.25) = $400.*

Make these adjustments car by car is not practical. *Using loss ratios adjusts for all class dimensions.* Using observed loss ratios instead of observed loss costs adjusts for driver age, driver sex, territory, vehicle types, and all other rating dimensions. We then restore the current rating relativities for the class dimensions that we are analyzing – male/female and urban/rural in this illustration.

The average observed loss costs for 1000 drivers in each of four classes are displayed above. The current relativities are 2.4 to 1 for male to female and 1.8 to 1 for urban to rural. The average SUV to sedan relativity is 1.2 to 1. SUV's comprise 40% of rural cars and 10% of urban cars. Unmarried males under the age of 21 comprise 10% of male drivers, and their average surcharge is 50%. Ideally, we would convert the observed loss costs to adjusted loss costs for the minimum bias analysis in the following manner.

- *Rural/female:* SUV's are 40% of rural cars, increasing the loss costs by a factor of 1 + 20% × 40% = 1.08. Were the cars all sedans, the observed loss costs would be reduced by a factor of 1/1.08 = 92.59%.
- *Urban/female:* The vehicle type factor is 1 + 20% × 10% = 1.02. Were the cars all sedans, the observed loss costs would be reduced by a factor of 1/1.02 = 98.04%.
- *Rural/male:* The vehicle type factor is 1 + 20% × 40% = 1.08 and the driver age factor is 1 + 10% × 50% = 1.05. Were the cars all sedans driven by adult drivers, the observed loss costs would be reduced by a factor of 1/(1.08 × 1.05) = 88.18%.
- *Urban/male:* The vehicle type factor is 1 + 20% × 10% = 1.02 and the driver age factor is 1 + 10% × 50% = 1.05. Were the cars all sedans driven by adult drivers, the observed loss costs would be reduced by a factor of 1/(1.02 × 1.05) = 93.37%.

We have made all the adjustments by the distribution of other class dimensions in the four cells of the matrix. This distribution is rarely available, and the procedure is complex when there are several class dimensions. A simple alternative is to divide the losses by the premium charged in each cell, and then multiply by the base rate times the current relativities for the two class dimensions which we are examining.

For each vehicle, we divide the losses by the premium, which is the base rate times the class relativities for all class dimensions. We multiply the result by the base rate times the class relativities for male/female and urban/rural. This is equivalent to dividing by the class relativities for the remaining dimensions.

The base pure premium for a female driving a sedan in a rural territory is $200. We work out the premium in each cell.

- Rural/female: We adjust for vehicle type with a multiplicative factor of 1 + 20% × 40% = 1.08. The average pure premium is $200 × 1.08 = $216.00.
- Urban/female: The vehicle type factor is 1 + 20% × 10% = 1.02. The average pure premium is $400 × 1.02 = $408.00.
- Rural/male: The vehicle type factor is 1 + 20% × 40% = 1.08 and the driver age factor is 1 + 10% × 50% = 1.05. The average pure premium is $500 × 1.08 × 1.05 = $567.00.
- Urban/male: The vehicle type factor is 1 + 20% × 10% = 1.02 and the driver age factor is 1 + 10% × 50% = 1.05. The average pure premium is $800 × 1.02 × 1.05 = $856.80.

The average loss cost in each cell divided by the average pure premium is the net loss ratio.[13]

	Urban	Rural
Male	$800 / $856.80 = 93.37%	$500 / $567.00 = 88.18%
Female	$400 / $408.00 = 98.04%	$200 / $216.00 = 92.59%

We have removed the effects of all class dimensions from the observed loss costs. We multiply by the current pure premium relativities for male/female and urban/rural to restore these effects to the observed data.

	Urban	Rural
Male	93.37% × 2.4 × 1.8 = 403.36%	88.18% × 2.4 × 1.0 = 211.64%
Female	98.04% × 1.0 × 1.8 = 176.47%	92.59% × 1.0 × 1.0 = 92.59%

For presentation purposes, we can reset the base class in each dimension to have a relativity of unity; these figures are relative loss ratios.

[13]These net loss ratios are net of expenses; they are losses divided by pure premiums. (Net loss ratios are sometimes called experience ratios.) In practice we have gross premiums, not pure premiums, so we use traditional loss ratios, not net loss ratios. The traditional loss ratios may be slightly distorted by expense flattening procedures. Loss costs show pure premium relativities, whereas traditional loss ratios show rate relativities. If the difference is material, offsetting adjustments must be made. On rate relativities versus pure premium relativities, see Feldblum [1996: PAP].

	Urban	Rural
Male	403.36% / 92.59% = 435.63%	211.64% / 92.59% = 228.57%
Female	176.47% / 92.59% = 190.59%	92.59% / 92.59% = 100.00%

If we wish, we can get adjusted loss costs by multiplying the cells by the base rate.

	Urban	Rural
Male	435.63% × $200 = $871.26	228.57% × $200 = $457.14
Female	190.59% × $200 = $381.18	100.00% × $200 = $200

EXERCISE: LOSS RATIO METHOD

The incurred losses and earned premium in each cell are shown below.

	Incurred Losses		*Earned Premium*	
	Urban	*Rural*	*Urban*	*Rural*
Male	$2,700	$2,000	$3,000	$4,000
Female	$1,500	$1,200	$2,400	$1,600

The current relativities by sex of driver and by garaging location are

Male: 1.50
Female: 1.00

Urban: 1.20
Rural: 1.00

Causes of Unequal Loss Ratios

To correct for potential distortions caused by an uneven distribution of insureds by other class dimensions, we use loss ratios instead of loss costs. If the rate relativities match the loss cost differences, the loss ratios should be equal in all cells, except for random loss fluctuations. In this example, the loss ratios are not all equal.

	Loss Ratios	
	Urban	*Rural*
Male	90.00%	50.00%
Female	62.50%	75.00%

There are several possible causes for the unequal loss ratios.

Cause 1 – Random loss fluctuations: Random loss fluctuations are a credibility issue. This paper assumes either that the data are fully credible or that the pricing actuary has already made (or will make) whatever adjustments are warranted by credibility considerations. Credibility adjustments for sparse data are an important actuarial issue, though they are beyond the scope of this paper.

Cause 2 – Improper rate relativities in other class dimensions combined with an uneven distribution of insureds by these other class dimensions: For example, perhaps the rates for a certain type of vehicle are too low, and the proportion of urban males driving that type of vehicle is greater than the proportions of the insureds in the other cells driving that type of vehicle.

If this is the cause of the differences, there is no perfect solution.[17] But if the distribution of insureds by the other class dimension is not too uneven, an inaccuracy in the rates will not distort our analysis too much. We may restate our assumption as follows:

> *For other class dimensions, either current rate relativities are accurate or the mix of insureds is relatively even across these other dimensions.*

In many instances, this assumption is not perfect. Nevertheless, even if the use of loss ratios does not perfectly correct for distortions caused by an uneven distribution of insureds along other class dimensions, it provides a partial correction.

Cause 3 – Inaccuracies in the rate relativities for the two class dimensions that we are examining (sex and territory): This is corrected by the minimum bias procedure, since the loss ratios by cell times the current relativities by cell equal the relative loss costs by cell.

> *Illustration 33.11:* Suppose the loss ratio for male drivers is 90% and the loss ratio for female drivers is 62.5%. If the current male to female rate relativity is 1.5 to 1, the male to female loss cost relativity is 1.5 × 90% to 1 × 62.5% = 2.16 to 1.

For the illustration in this section, we form a matrix of relativities by sex and territory:

Current Rate Relativities	*Urban*	*Rural*
Male	1.80	1.50
Female	1.20	1.00

The relative loss costs by sex and territory are the product of the relativities and the loss ratios:

Loss Cost Relativities	*Urban*	*Rural*
Male	1.62	0.75
Female	0.75	0.75

We can now determine optimal rate relativities by any of the minimum bias models discussed in this paper.

Cause 4 – Improper Model Specification: We may be using a multiplicative model when an additive model would be more proper (or vice versa). Sometimes neither a multiplicative nor an additive model is ideal. The choice of model is discussed in its own section.

The squared error and χ^2 bias functions minimize squared residuals.

We examine other bias functions, beginning with squared error and χ^2 functions applied to additive and multiplicative models for the 2 by 2 illustration above. We examine two goodness-of-fit tests (average absolute error and χ^2) the relation between the bias function and the goodness-of-fit test. We then discuss maximum likelihood bias functions and the distributions commonly used with them.

Illustration 33.3: The 2 × 2 illustration with a multiplicative model is reproduced below.

	Urban	*Rural*		*$Terr_1$*	*$Terr_2$*
Male	\$800	\$500	sex_1	$200 \times s_1 \times t_1$	$200 \times s_1 \times t_2$
Female	\$400	\$200	sex_2	$200 \times s_2 \times t_1$	$200 \times s_2 \times t_2$

The left-hand side shows observed loss costs; the right-hand side shows indicated pure premiums. We pick class relativities so that the indicated pure premiums are as close as possible to the observed loss costs. We fit the class relativities using one of the methods below.

1. Minimize the average absolute error between the indicated and observed figures
2. Minimize the sum of the squared errors between the indicated and observed figures (the least squares bias function)
3. Minimize the sum of the relative squared errors between the indicated and observed figures (i.e., minimize the χ-squared error)
4. Maximize the likelihood of obtaining the observations given the class relativities

Minimizing the average absolute error makes sense to practitioners, but it is rarely used in statistics, because it is less mathematically tractable.[18] We use the average absolute error as one of the goodness-of-fit tests. Given a set of class relativities, it is easy to calculate the average absolute error; it is harder to determine the class relativities that minimize the average absolute error. The three other methods result in iterative equations for the minimum bias procedure. We show the procedures, and then discuss the intuition for each.

The squared error for each cell is the square of the difference between the observed loss costs and the indicated pure premium. For urban male drivers, this is $(\$800 - \$200 \times s_1 \times t_1)^2$.

We sum the squared errors for the four cells to get the sum of squared errors (SSE):

$$\begin{aligned} SSE &= (\$800 - \$200 \times s_1 \times t_1)^2 && \text{urban male} \\ &+ (\$500 - \$200 \times s_1 \times t_2)^2 && \text{rural male} \\ &+ (\$400 - \$200 \times s_2 \times t_1)^2 && \text{urban female} \\ &+ (\$200 - \$200 \times s_2 \times t_2)^2 && \text{rural female} \end{aligned}$$

To minimize the sum of the squared errors, we set the partial derivatives with respect to each variable equal to zero. For the *male* class relativity (s_1), we have

$$\begin{aligned} 0 = \partial SSE/\partial s_1 &= 2 \times (\$800 - \$200 \times s_1 \times t_1) \times (-\$200 \times t_1) \\ &+ 2 \times (\$500 - \$200 \times s_1 \times t_2) \times (-\$200 \times t_2) \end{aligned}$$

We consider the cells only in the male (s_1) row. The other cells do not have an s_1 term in the squared error, so the partial derivative with respect to s_1 is zero.

Taking partial derivatives with respect to each of the class relativities gives four equations in four unknowns. The equations are not linear, so we solve them iteratively. We can choose the same starting values for the squared error bias function as we chose for the balance principle (namely $t_1 = 2$ and $t_2 = 1$):

	Urban	Rural		$Terr_1 = 2$	$Terr_2 = 1$
Male	\$800	\$500	sex_1	$200 \times s_1 \times 2$	$200 \times s_1 \times 1$
Female	\$400	\$200	sex_2	$200 \times s_2 \times 2$	$200 \times s_2 \times 1$

Using the squared error bias function, we solve for the male relativity s_1. To avoid extra zeros, we choose a base pure premium of \$2 and we evaluate the observed pure premiums in multiples of \$100.

$$0 = \partial SSE/\partial s_1 = 2 \times (\$8 - \$2 \times s_1 \times 2) \times (-\$2 \times 2) + 2 \times (\$5 - \$2 \times s_1 \times 1) \times (-\$2 \times 1)$$

$$-64 + 32s_1 - 20 + 8s_1 = 0$$
$$40s_1 = 84$$
$$s_1 = 2.1$$

Similarly, we solve for the female relativity (s_2):

$$0 = \partial SSE/\partial s_2 = 2 \times (\$4 - \$2 \times s_2 \times 2) \times (-\$2 \times 2) + 2 \times (\$2 - \$2 \times s_2 \times 1) \times (-\$2 \times 1)$$

$$-32 + 32s_2 - 8 + 8s_2 = 0$$
$$-40 + 40s_2 = 0$$
$$s_2 = 1$$

We now discard the starting values of $t_1 = 2$ and $t_2 = 1$. Using the intermediate values of $s_1 = 2.1$ and $s_2 = 1$, we set the partial derivatives of the sum of the squared errors with respect to t_1 and t_2 equal to zero and we solve for new values of t_1 and t_2. We continue in this fashion until the series converges.

Iteration	Urban	Rural	Male	Female
Initial	2.0000	1.0000		
1-a			2.1000	1.0000
1-b	1.9224	1.1553		
2-a			2.1029	0.9940
2-b	1.9223	1.1555		
Normalized	1.6636	1.0000	2.1155	1.0000
Normalized Base Pure Premium			\$200.00 × 1.1555 × 0.9940 = \$229.71	

MULTIPLE OBSERVATIONS IN EACH CELL

If the number of observations differs by cell, we treat each observation as a separate cell; that is, we weight the mean value in the cell by the number of observations in the cell.

To illustrate, suppose the cells have 19, 17, 13, and 11 observations, as shown below:

	Urban	*Rural*		$terr_1 = 2$	$terr_2 = 1$
Male	\$800	\$500	sex_1	19	17
Female	\$400	\$200	sex_2	13	11

Using the squared error bias function, we solve for the male relativity s_1. To avoid extra zeros, we choose a base pure premium of \$2 and we evaluate the observed pure premiums in multiples of \$100.

$$0 = \partial SSE/\partial s_1 = 19 \times 2 \times (\$8 - \$2 \times s_1 \times 2) \times (-\$2 \times 2) + 17 \times 2 \times (\$5 - \$2 \times s_1 \times 1) \times (-\$2 \times 1)$$

$$-19 \times 64 + 19 \times 32s_1 - 17 \times 20 + 17 \times 8s_1 = 0$$
$$(19 \times 32 + 17 \times 8)s_1 = (19 \times 64 + 17 \times 20)$$
$$s_1 = (19 \times 64 + 17 \times 20) / (19 \times 32 + 17 \times 8) = 2.091398$$

Similarly, we solve for the female relativity (s_2):

$$0 = \partial SSE/\partial s_2 = 13 \times 2 \times (\$4 - \$2 \times s_2 \times 2) \times (-\$2 \times 2) + 11 \times 2 \times (\$2 - \$2 \times s_2 \times 1) \times (-\$2 \times 1)$$

$$-13 \times 32 + 13 \times 32s_2 - 11 \times 8 + 11 \times 8s_2 = 0$$
$$-(13 \times 32 + 11 \times 8)s_2 = (13 \times 32 + 11 \times 8)$$
$$s_2 = 1$$

Squared error and χ-squared intuition

The bias function makes a difference, even in this simple illustration: the male/female relativity is 2.1667 using the balance principle and 2.1155 using the squared error bias function.

The balance principle ensures that the total error in each class dimension is zero. The squared-error bias function minimizes the aggregate squared error, and the χ-squared bias function minimizes the aggregate squared error as percentages of expected values. The squared-error and χ^2 bias functions place more weight on outliers, where the squares of the errors are large. The balance principle places more weight on cells with large dollar values.

Illustration 33.4: A class system with two dimensions has male vs female in one dimension and territories 1, 2, and 3 in the other dimension. The starting relativities are 1.00, 2.00, and 3.00 for territories 1, 2, and 3. The observed loss costs for the three territories in the male row are $2, $4, and $12, with equal exposures in each cell. We assume a base rate of $1.00.

	Territory 1 (1.00)	*Territory 2 (2.00)*	*Territory 3 (3.00)*
Male	$2.00	$4.00	$12.00
Female	—	—	—

We want to determine the indicated relativity for males. Our concern here is not to solve this problem but to understand the effects of the different bias functions.

- If the male relativity is 2.00, the indicated pure premiums are $2, $4, and $6. The first two cells have a perfect fit, and the third cell is too low by $6.
- If the male relativity is 4.00, the indicated pure premiums are $4, $8, and $12. The first two cells are too high by a total of $6, and the third cell has a perfect fit.

The balance principle considers the first power of the errors. The average observed loss cost is ($2 + $4 + $12)/3 = $6.00. The average territory relativity is 2.00. To achieve balance, we choose a male relativity of 3.00. The indicated pure premiums are $3, $6, and $9. The first two cells are too high by a total of $3, and the third cell is too low by $3. The indicated male/female relativity is $6 / $2 = 3.00.

If we optimize with the balance principle, the sum of the squared errors is $(3 - 2)^2 + (6 - 4)^2 + (9 - 12)^2 = 14$. The squared error bias function is more concerned with the large error in territory 3 than with the small errors in territories 1 and 2. To minimize the sum of squared errors, we increase the male relativity slightly, reducing the error in territory 3 and increasing the errors in territories 1 and 2. To solve the problem using a squared error bias function, we minimize the sum of squared errors:

$$SSE = (2 - x)^2 + (4 - 2x)^2 + (12 - 3x)^2.$$

Setting the partial derivative with respect to *x* equal to zero gives

$$\partial SSE/\partial x = 2(2-x)(-1) + 2(4-2x)(-2) + 2(12-3x)(-3) = 0$$

$$4 + 16 + 72 = 2x + 8x + 18x$$
$$92 = 28x$$
$$x = 92 / 28 = 3.286$$

The sum of the squared errors is $(3.286 - 2)^2 + (6.571 - 4)^2 + (9.857 - 12)^2 = 12.857$, which is less than the squared error of 14 under the balance principle. Minimizing the sum of the squared errors yields 3.286, not the average, which is 3.00.

SQUARED ERROR MINIMIZATION

The illustration above seems odd to some statisticians. We are choosing a value to minimize the squared error among a series of observations. An elementary statistical theorem is that the average minimizes the sum of the squared errors. This seems inconsistent with the comments above.

When we set rates in a single dimension class system, minimizing the squared error produces the arithmetic average. The following illustration explains this.

Illustration: We measure a patient's fever with an old, imprecise thermometer. The thermometer is unbiased, but the observed readings are distorted by sampling error. We perform nine trials, and we observe readings of (100.1, 100.2, . . . , 100.9). (The readings were not in this order, so there is no trend; we have simply arranged them in ascending numerical order.) Using the least squared error function, we determine the best estimate of the patient's temperature.

We rephrase the illustration mathematically. We have observed values of $z_1, z_2, \ldots, z_n$, and we must choose a single value – call it z^* – to minimize the squared error.

The sum of the squared errors is $\sum (z_i - z^*)^2$. The partial derivative of this sum with respect to z^* is $\sum 2(z_i - z^*)(-1)$. Setting this equal to zero gives $z^* = \sum z_i \div n$. The indicated z^* is the average of the z_i's.

In the temperature measurement illustration, the average of the nine observations is 100.5. This is the solution using the squared error bias function.

If we had chosen instead some other value, such as 100.3, we could correct this estimate by the average of the errors. The error in each observation is the observation minus 100.3. This is the series (–0.02, –0.01, 0, +0.01, . . . , +0.06). The average is +0.02. The corrected estimate is 100.3 + 0.02 = 100.5.

This is not true for multi-dimensional systems. In a multiplicative model with two dimensions, the z_i's are the observed values. The z^* is the indicated relativity for one of the two dimensions. The other dimension has relativities of $y_1, y_2, \ldots, y_m$.

The sum of the squared errors is $\sum\sum (z_i - y_j \times z^*)^2$. The partial derivative of this sum with respect to z^* is $\sum\sum 2(z_i - y_j \times z^*)(-y_j)$. Setting this equal to zero gives $z^* = \sum\sum z_i \div \sum y_j^2$.

The indicated z* is no longer the average of the z_i's. Rather, this result is the solution to the minimum bias procedure using the squared error bias function, as we show next.

Balance Principle Optimization

When we seek a pure premium for one dimension, minimizing the squared error produces the arithmetic average. With two or more dimensions, the balance principle selects the multi-dimensional equivalent to the mean of each class across the other dimension(s).

The balance principle provides an unbiased solution; Bailey [1963] considers it the only unbiased solution (see below). Some actuaries believe that an unbiased solution is more likely to maximize the firm's profitability than a biased solution.[19]

GENERAL SQUARE ERROR MINIMIZATION, MULTIPLICATIVE MODEL

We consider a more general two dimensional class system. The base pure premium is *B*. We again assume one exposure per cell (or the same number of exposures per cell) to keep the equations simple. In practice, one must multiply all terms by the number of exposures.

Suppose we have two dimensions, age of driver and territory, with *n* age classes and *m* territories. The observed loss cost in the i^{th} age class and the j^{th} territory is r_{ij}. The indicated pure premium in the i^{th} age class and the j^{th} territory is $B \times x_i \times y_j$.

The squared error in any cell is $(r_{ij} - Bx_iy_j)^2$. The sum of the squared errors is

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{m} (r_{ij} - Bx_i y_j)^2$$

We take partial derivatives with respect to each variable and set them equal to zero. The (n+m) equations are not linear, so we must search for a solution by numerical methods. We choose starting values for one dimension – say, the y_j's. To solve for the value of x_1, we take the partial derivative with respect to x_1 and set it equal to zero:

$$\sum_{J=1}^{m} 2(r_{1j} - Bx_1 y_j)(-By_j) = 0$$

This gives

$$x_1 = \sum_{j=1}^{m} (r_{1j} \times y_j) - \sum_{j=1}^{m} By_j^2$$

The x_1 is a variable. The *y* values are fixed; they are not variables once we have assigned starting values to the *y* values.

We repeat this procedure to solve for $x_2, x_3, \ldots, x_n$. We then discard the starting *y* values and solve for new values of the *y* variables using the same procedure as we used for the *x* variables.

We have (n+m) variables, and we have (n+m) equations. The constraints for least squares minimization are the same as the constraints for the balance principle. There is one totality constraint, since taking the sum of the squared errors along the rows is the same as taking the sum of the squared errors along the columns.

This means that we have only (n+m–1) independent equations. In addition, we could multiply all the relativities along any dimension by a constant and divide the base pure premium by the same constant.

SQUARED ERROR MINIMIZATION, ADDITIVE MODEL

We can also use an additive model with the least squares bias function. We first show the results for the elementary 2 by 2 illustration. Below are the same observed loss costs and indicated pure premiums we have been using.

	Urban	Rural		$terr_1$	$terr_2$
Male	$800	$500	sex_1	$200 + s_1 + t_1$	$200 + s_1 + t_2$
Female	$400	$200	sex_2	$200 + s_2 + t_1$	$200 + s_2 + t_2$

As mentioned earlier, there are three mathematically equivalent ways of defining the additive model; the solution method is the same for each of them. The pure premium in cell x_iy_j is

A. Base pure premium + $x_i + y_j$,
B. Base pure premium × $(1 + u_i + v_j)$, or
C. Base pure premium × $(p_i + q_j)$

We use the first of these three equations here for its intuitive simplicity. Note that a multiplicative relationship between the base pure premium and the relativities does not make the model multiplicative. If the relationship among the factors is additive, the model is additive. A combined multiplicative and additive model has relationships among the relativities that are both multiplicative and additive.

The method used here is the same as the method used for the multiplicative model above. For the male urban cell, the squared error is $(\$800 - \$200 - s_1 - t_1)^2$. The sum of the squared errors for all four cells is

$$\begin{aligned} Q &= (\$800 - \$200 - s_1 - t_1)^2 + (\$500 - \$200 - s_1 - t_2)^2 \\ &+ (\$400 - \$200 - s_2 - t_1)^2 + (\$200 - \$200 - s_2 - t_2)^2 \end{aligned}$$

We take partial derivatives with respect to each variable and set them equal to zero. The partial derivative with respect to s_1 is

$$\partial Q/\partial s_1 = 2(\$800 - \$200 - s_1 - t_1)(-1) + 2(\$500 - \$200 - s_1 - t_2)(-1) = 0.$$

or

$$s_1 = (\$900 - t_1 - t_2) \div 2.$$

For the additive model with the least squares bias function, the simultaneous equations are linear, and we can solve them directly. Nevertheless, it is easier to program the solution using numerical methods. If we choose starting values of t_1 = \$250 and t_2 = \$0, we get s_1 = \$325. We leave it to the reader to verify that the relativities converge to the same figures as the additive model with the balance principle.

GENERAL SQUARED ERROR MINIMIZATION, ADDITIVE MODEL

For the general formula, we let B = the base pure premium. The sum of the squared errors is

$$SSE = \sum_{i=1}^{n} \sum_{j=1}^{m} (r_{ij} - B - x_i - y_j)^2$$

We take the partial derivative with respect to x_i and set it equal to zero:

$$\frac{\partial SSE}{\partial x_i} = \sum_{j=1}^{m} 2(r_{1j} - B - x_1 - y_j)(-1) = 0$$

or

$$x_i = \sum_{j=1}^{m} (r_{ij} - y_j) / m - B$$

Maximum likelihood bias function.

Some statisticians prefer a maximum likelihood bias function when fitting a distribution to observed data. The maximum likelihood bias function requires a probability distribution for the values in a class. The distribution of loss costs is not a simple parametric distribution, such as a Gamma or Poisson distribution, so a pricing actuary may forecast claim frequency and severity separately, with a Poisson distribution for frequency and a Gamma distribution for severity.

For simplicity, we use an exponential distribution: the likelihood of a loss of size *x* is $\lambda e^{-\lambda x}$. An exponential distribution is too skewed for practical application, but it illustrates clearly the maximum likelihood method. We use integration by parts to solve for the mean of the exponential distribution function:

$$\int_0^\infty x\lambda e^{-\lambda x} dx = (-x\lambda e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x}) \Big|_0^\infty = \frac{1}{\lambda}$$

To fit an exponential distribution to a set of insurance losses with multiple class dimensions, we determine λ by maximum likelihood.

Illustration: Suppose we observe five claims, with sizes of $3,000, $5,000, $15,000, $20,000, and $80,000. For a given value of λ, the likelihood of a loss equal to $3,000 is $\lambda e^{-\lambda \times 3{,}000}$. The likelihood (L) of the five losses listed above is the product of the likelihoods of each loss, or

$$L = \lambda e^{-\lambda \times 3{,}000} \times \lambda e^{-\lambda \times 5{,}000} \times \lambda e^{-\lambda \times 15{,}000} \times \lambda e^{-\lambda \times 20{,}000} \times \lambda e^{-\lambda \times 80{,}000}.$$

The likelihood equals $\lambda^5 e^{-\lambda \times 123{,}000}$, where $123,000 is the sum of the losses. To find the λ with the greatest likelihood, we set the derivative with respect to λ equal to zero. Maximizing a strictly increasing function, like the likelihood function, is the same as maximizing its logarithm. The logarithm of the likelihood (the log-likelihood, or LL) is

$$LL = \ln L = 5 \ln \lambda – 123{,}000 \times \lambda$$
$$d (\ln L)/d\lambda = 5/\lambda – 123{,}000 = 0, \text{ or } \lambda = 5/123{,}000$$

For a single class dimension, the maximum likelihood estimator for an exponential distribution is the mean. For a class system with two or more dimensions, the maximum likelihood estimators is not always the mean.

Using maximum likelihood for the minimum bias procedure

The rating model uses the class relativities to determine the expected loss in each cell. The maximum likelihood test is most practicable when a single parameter distribution is used and the mean of the distribution equals the parameter itself or some simple function of the parameter, such as its reciprocal. It is most valuable when the distribution is a reasonable reflection of the insurance process.

The exponential and Poisson distributions have these properties. We illustrate a multiplicative model with the exponential distribution function, using the same illustration as before.

	Urban	Rural		$Terr_1$	$Terr_2$
Male	\$800	\$500	sex_1	$200 \times s_1 \times t_1$	$200 \times s_1 \times t_2$
Female	\$400	\$200	sex_2	$200 \times s_2 \times t_1$	$200 \times s_2 \times t_2$

Each class has an assumed exponential distribution of loss costs. If the indicated pure premium is \$200, we expect the observed losses to follow an exponential distribution with a mean of \$200. The λ differs by cell. The indicated pure premium in that cell is $1/\lambda$.

For the urban/male cell, the loss costs have an exponential distribution with the parameter λ equal to $1/(\$200 \times s_1 \times t_1)$. We choose starting values for $t_1 = 2.00$ and $t_2 = 1.00$. We determine the likelihood of the observed loss costs. The value of λ for the urban/male cell is $1/(200 \times s_1 \times t_1) = 1/(400 \times s_1)$. The likelihood of the \$800 loss cost in the urban/male cell is

$$\frac{1}{400s_1} e^{\frac{-800}{400s_1}} = \frac{1}{400s_1} e^{\frac{-2}{s_1}}$$

The likelihoods of the observed values in the other cells are determined in the same manner. For simplicity, we maximize the log-likelihood:

- The likelihood of four observed values is the *product* of the four likelihoods.
- The log-likelihood of four observed values is the *sum* of the four log-likelihoods.

The partial derivative of the log-likelihood with respect to s_1 depends on the log-likelihoods in the male row only. This is the same simplification that we used for the least squares method and the χ-squared method.

The total log-likelihood in the male row is $LL = -\ln(400s_1) - (800/400) \times 1/s_1 - \ln(200s_1) - (500/200) \times 1/s_1$. The partial derivative with respect to s_1 is $-1/s_1 + 2s_{1-2} - 1/s_1 + 2.5s_{1-2}$. We set this equal to zero:

$$\partial LL/\partial s_1 = -1/s_1 + 2s_{1-2} - 1/s_1 + 2.5s_{1-2} = 0$$
$$-s_1 + 2 - s_1 + 2.5 = 0, \text{ because } s_1 \neq 0$$
$$s_1 = 2.25.$$

Similarly, the likelihood of the \$400 loss cost in the urban/female cell is $1/400s_2 \times \exp(-400/400s_2) = 1/(400s_2) \times \exp(-1/s_2)$, and the likelihood of the \$200 loss cost in the rural/female cell is $1/200s_2 \times \exp(-200/200s_2) = 1/(200s_2) \times \exp(-1/s_2)$. The total log-likelihood of the values in the female row is

$$LL = -\ln(400s_2) - (400/400) \times 1/s_2 - \ln(200s_2) - (200/200) \times 1/s_2$$

The partial derivative with respect to s_2 is $-1/s_2 + 1s_{2-2} - 1/s_2 + 1s_{2-2}$. We set this equal to zero.

$$\partial LL/\partial s_2 = -1/s_2 + 1s_{2-2} - 1/s_2 + 1s_{2-2} = 0$$
$$-s_2 + 1 - s_2 + 1 = 0$$
$$s_2 = 1.00.$$

The series converges to the following relativities.

Iteration	Urban	Rural	Male	Female
Initial	2.0000	1.0000		
1-a			2.2500	1.0000

1-b	1.8889	1.0556		
2-a			2.2430	1.0031
2-b	1.8886	1.0557		
3-a			2.2430	1.0031
Normalized	1.7889	1.0000	2.2361	1.0000
Normalized Base Pure Premium			$200.00 × 1.0557 × 1.0031 = $211.80	

If the distribution of loss costs is Poisson or exponential, we derive simple recursive equation. In practice, we don't always know the distributions. The Poisson distribution is reasonable for loss frequency distributions, though not for loss severity distributions. Pricing actuaries often examine claim frequency relativities by class to judge rate adequacy, using generalized linear models or minimum bias procedures.

THE BIAS FUNCTION IS BASED ON MATHEMATICAL TRACTABILITY, SOCIAL EQUITY, AND ECONOMIC OPTIMIZATION.

The optimal class relativities depend on the choice of bias function, based on mathematical tractability, social equity, and economic optimization.

Mathematical tractability was of concern when computer power was limited: some bias functions gave simple relations while other bias functions gave intractable equations. With modern spreadsheets, tractability issues are minor. The balance principle solves for the mean, the average absolute error solves for the median. It is not uncommon for actuaries to use the median instead of the mean in practical problems.

Social equity is subjective, though it is vital to the success of a regulated industry like insurance. The balance principle sometimes results in large errors for outlying, high-rated cells. If a multiplicative model is used when an additive model is more appropriate, the errors for outlying cells are frequently overcharges.

Economic optimization drives the behavior of firms in free markets, who seek to maximize profits and minimize losses. Suppose an insurer issues three policies and must choose between two rating systems.

Under rating system A, it expects to lose $1.00 each on the two policies and break even on the third policy. Under rating system B, it expects to break even on the first two policies and to lose $1.50 on the third policy.

Rating system A is off by $2.00 using the balance principle while rating system B is off by $1.50. Using the squared error bias function, rating system A is off by 2 dollars-squared while rating system B is off by 2.25 dollars-squared. The balance principle says we should choose rating system B, and the squared error bias function says we should choose rating system A.

To maximize profits, we prefer rating system B. In practice, economic forces are more complex than short-term profit maximization. There are many reasons for avoiding errors, including consumer dissatisfaction and public relations. In democratic systems where social opinion and political pressures are strong, firms may sacrifice short-term profit maximization to achieve other ends; objectives such as workforce diversity and environmental protection are examples. Furthermore, manager incentives may encourage the pursuit of other goals, such as corporate growth instead of profit maximization. Nevertheless, profit maximization remains the dominant corporate goal. The pricing actuary should keep these social and economic considerations in mind when choosing a bias function for the minimum bias procedure.

Summary

Each model discussed here uses iterative functions. The pricing actuary determines a rating function – such as multiplicative, additive, or combined – and a bias function (balance principle, least squares, χ-squared, or maximum likelihood). For the maximum likelihood bias function, the actuary selects a probability distribution function for the loss costs (or other values) in each cell.

The type of data in each cell is generally either loss costs or loss ratios. If one uses all the dimensions of the class system in the analysis, it is easier to use loss costs. If some class dimensions are not included, and if the distribution of exposures is not even among these other class dimensions, loss ratios may be used.

We list here the models and their recursive equations.

Multiplicative model, balance principle:

$$x_i = \frac{\sum_j n_{ij} r_{ij}}{\sum_j n_{ij} y_j}$$

Additive model, balance principle:

$$x_i = \frac{\sum_j n_{ij}(r_{ij} - y_j)}{\sum_j n_{ij}}$$

Multiplicative model, least squares:

$$x_i = \sum (n_{ij}^2 \times r_{ij} \times y_j) \div \sum (n_{ij}^2 \times y_j^2).$$

Additive model, least squares:

$$x_i = \frac{\sum n_{ij}^2 \times (r_{ij} - y_j)}{\sum n_{ij}^2} - B$$

Multiplicative model, χ-squared:

$$x_i = [\sum (n_{ij} \times r_{ij}^2 / y_j) / \sum n_{ij}\, y_j]^{0.5}$$

Additive model, χ-squared:

$$\Delta x_i = \frac{\sum_j n_{i,j}\left(\frac{r_{i,j}}{x_i + y_j}\right)^2 - \sum_j n_{i,j}}{2\sum_j n_{i,j}\left(\frac{r_{i,j}}{x_i + y_j}\right)^2\left(\frac{1}{x_i + y_j}\right)}$$

Multiplicative model, maximum likelihood, normal density function:

$$x_i = \frac{\sum_j n_{ij}^2 r_{ij} y_j}{\sum_j n_{ij}^2 y_j^2}$$

Additive model, maximum likelihood, normal density function:

$$x_i = \frac{\sum_j n_{ij}^2 (r_{ij} - y_j)}{\sum_j n_{ij}^2}$$

Multiplicative model, maximum likelihood, exponential density function:

$$x_i = \frac{\sum_j \frac{r_{ij}}{y_j}}{k}$$

where “k” is the number of classes in the “j” dimension.

The recursive functions for a multiplicative model, maximum likelihood, Poisson distribution function are the same as those for the multiplicative model, balance principle.

ENDNOTES

1 Unequal cell populations are discussed later.

2 This assumption is rarely tested, and the independence of class dimensions does not necessarily imply a multiplicative model. Neither the additive nor the multiplicative model is perfect, but the multiplicative model is usually better.

3 Life insurance rating systems provide an example. If smokers have twice the mortality of non-smokers, and persons with high-blood pressure have twice the mortality of persons with average blood pressure, should high-blood pressure smokers have four times the mortality of average blood pressure non-smokers? Life insurance underwriters employ judgment to assess the rating for applicants with multiple causes of high mortality. A pure multiplicative rating system would not be appropriate.

4 The bias function is not a standard statistical term, and the balance principle is not a standard principle. The bias function determines how close the indicated pure premiums are to the observed loss costs or how great the mismatch is between these two sets of data. The sum of the squared deviations and the χ-squared deviation are common statistical bias functions. The balance principle,

introduced by Bailey and Simon in 1960 and endorsed again by Bailey in 1963, minimizes the bias along the dimensions of the class system, leading to the term *minimum bias*.

[5] To keep the notation simple, the rating dimensions are male vs female and urban vs rural. The formulas in the illustrations use $sex_1 = s_1$ = male, $sex_2 = s_2$ = female, $territory_1 = t_1$ = urban, and $territory_2 = t_2$ = rural. The recursive equations use variable names of x, y, and z, and rating dimensions of i and j.

[6] With so much leeway in the class relativities, one might ask what we are optimizing. We are optimizing the indicated pure premiums. Each set of class relativities gives the same indicated pure premiums. The optimization is relative to the bias function. The optimal pure premiums have the least bias, the least squared error, the least χ-squared value, or the maximum likelihood, depending on the bias function.

[7] In practice, formulas B or C might be preferred, since only the base pure premium need be increased for inflation. In formula A, the base pure premium and all the relativities must be increased for inflation.

[8] Companies may do this for marketing reasons. All drivers get discounts from the base pure premium, so all drivers feel they are gaining from the class system.

[9] In most cases, the direction of the bias does not depend on the type of rating model. The more common scenario might show an observed loss cost of $600, an additive model indicated pure premium of $550, and a multiplicative model indicated pure premium of $530. We might infer that the random loss fluctuations underlying these cells values have had a net positive effect. For very high rated or very low rated classes, the multiplicative and additive models often give opposite results, as is the case here.

[10] In the illustrations, we use s for the row relativity and t for the column relativity as abbreviations for the class dimensions (sex and territory). The variables x and y are commonly used in the literature.

[11] We sum over the j subscript when we balance along the rows (the i subscripts). We do this separately for each i. When we balance along the columns, we sum over the i subscripts separately for each j.

[12] To optimize this rating model, the balance principle is not sufficient; we would have to employ one of the other bias functions. The balance principle provides $i+j$ equations, but we have $i+2j$ variables. The other bias functions discussed in this paper provide $i+2j$ equations.

[13] Bailey and Simon [1960] assume that if all claims are independent, the variance is proportional to the number of claims, so the standard deviation is proportional to the square root of the number of claims. See also Longley-Cook, Lawrence H., "An Introduction to Credibility Theory," *Proceedings of the Casualty Actuarial Society*, Volume 49 (1962), pages 194-221. After the writings of Hans Bühlmann, Gary Venter, Howard Mahler, and others, this is no longer the standard rationale for credibility.

[14] The countrywide analysis may actually be done on all tort liability states or all no-fault states, since the bodily injury rate relativities may be higher for SUV's (sports utility vehicles) than for sedans in tort liability states, whereas the reverse may be true in no-fault states.

[15] In practice, we use loss ratios adjusted to the base rates for the classification dimensions included in the minimum bias analysis, though this is not shown in the illustration.

[16] This section assumes that the pure premium relativities are the same as the rate relativities.

[17] Without information about the other classification dimensions, we can not optimize the class system.

[18] See, however, Cook [1967], page 200: "Why then do we use the method of least squares? Simply because absolute values are alleged to be mathematically inconvenient." Cook provides an algorithm for minimizing the average absolute error, which is simple to compute and even easier to program. [Charles Cook, "The Minimum Absolute Deviation Trend Line," Proceedings of the CAS, vol 80 [1967], pages 200-204.]

[19] There are exceptional scenarios when a different bias function may be better. In a jurisdiction that places restrictions on risk classification, the bias function may have to be changed to accommodate these restrictions. If the insurer seeks to expand in certain classifications for competitive or marketing reasons, the minimum bias procedure may not accommodate the insurer's strategy. In most scenarios, however, the balance principle serves the economic interests of the firm.