

## Modules 3 and 4: Solow growth model practice problems

### *Practice problems for the final exam*

(The attached PDF file has better formatting.)

This posting gives sample final exam problems for the Solow growth model, which is explained in Chapters 3 and 4 of Barro's textbook. The final exam problems are multiple choice; these practice problems are essay questions so that the solutions can be better explained.

#### \*\* Question 4.1: Patterns in growth rates

An economy follows the Solow growth model with no technological progress.

In 20X2, the expected growth rate of income per worker is 3% and the population growth rate is  $n$ .

What is the expected growth rate of income per worker in 20X3?

- A. 3%
- B. More than 3%
- C. Less than 3%
- D. More than 3% if  $n > 0$ ; less than 3% if  $n < 0$
- E. Less than 3% if  $n > 0$ ; more than 3% if  $n < 0$

Answer 4.1: C

Capital has decreasing marginal utility, so the growth rate of income per worker slows as capital increases.

*Question:* Shouldn't this say: the growth rate of income per worker slows as *income per worker* increases?

*Answer:* If the Solow growth model with no technological progress,  $A$  does not change. In the steady state, the growth rate of income per worker is zero. If the expected growth rate of Income per worker is positive, the economy is in the transition phase with  $k_0 < k^*$ .

Income (real GDP) changes in capital or labor change. Income per worker changes if capital per worker changes. If income per worker increases 3% per annum, capital per worker must be increasing. The marginal utility of capital decreases, so the growth rate of capital per worker decreases. The growth rate of income per worker is  $\alpha \times$  the growth rate of capital per worker, where  $\alpha$  is the parameter of the (Cobb-Douglas) production function.  $\alpha$  is a constant, so the growth rate of income per worker has the same pattern as the growth rate of capital per worker.

See Barro, *Macroeconomics*, Chapter 3, economic growth, page 43

**\*\* Exercise 4.2: Solow growth model**

An economy follows a Solow growth model with no technological progress. In equilibrium

$y^*$  = steady state income per worker = 25 units per worker per annum

$\delta$  = depreciation rate of capital = 5% per annum

$k^*$  = steady state capital per worker = 100 units per worker

$n$  = population growth rate = 2% per annum

What is the savings rate in equilibrium?

Solution 4.2: Equation 3.16 on page 43, column 1, says:  $\Delta k/k = s \times (y/k) - s\delta - n$ .

In equilibrium, the steady state growth rate of capital per worker is zero, so  $s \times (y/k) - s\delta - n = 0 \Rightarrow$

$$s \times (25 \text{ units} / 100 \text{ units}) - s \times 5\% - 2\% = 0$$

$$s \times (25\% - 5\%) = 2\%$$

$$s = 2\% / 20\% = 10\%$$

$s$  = savings rate = 10%

*Question:* What are the units of income and capital?

*Answer:* The units don't matter. They could be real (deflated) dollars or euros or British pounds, or they could be goods (loaves of bread or flasks of wine). The Solow growth model has  $y^* / k^*$ , so the units cancel out.

See Barro, *Macroeconomics*, Chapter 3, "Economic Growth," page 43

**\*\* Exercise 4.3: Solow growth model**

An economy follows a Solow growth model with no technological progress, with  $k_0 < k^*$  in the transition phase.

- A. How does the savings rate affect the growth rate of capital per worker in the transition phase?
- B. How does the technology level affect the growth rate of capital per worker in the transition phase?
- C. How does the depreciation rate affect the growth rate of capital per worker in the transition phase?
- D. How does the population growth rate affect the growth rate of capital per worker in the transition phase?

*Part A:* Equation 3.16 on page 43, column 1, says:  $\Delta k/k = s \times (y/k) - s\delta - n$ .

If  $y/k > \delta$ , a higher savings rate increases the growth rate of capital per worker.

More savings by households means more investment in capital by business. The money saved by households is used to buy bonds. The businesses issuing the bonds use the money to build factories and buy equipment, so capital per worker increases.

*Question:* Where do we see this in practice?

*Answer:* Several East Asian countries have high savings rates. These East Asian countries grew rapidly, as the savings increased capital per worker and income per worker. Some southern European countries, like Greece, Italy, and Spain have low savings rates, and their economies have grown more slowly.

*Part B:* The technology level affects the ratio of  $y$  to  $k$  (income to capital):  $y = A \times f(k)$ . A higher technology level raises  $y/k$  and raises the growth rate of capital per worker.

*Question:* Where do we see this in practice?

*Answer:* Israel has a high technology level, because of high investment in education, a national culture of long education, strong protection of property rights, and an open economy. It has a high growth rate of capital per worker, despite a lack of natural resources. In contrast, some other Middle Eastern countries (Saudi Arabia, Egypt, Iraq) have low technology levels, for several reasons:

- Their traditional religious culture does not value secular (scientific) study.
- Abundant natural resources (oil) in some countries lead them to ignore technological progress.
- Autocratic governments and closed economies don't allow much diffusion of knowledge.

*Part C:* A higher depreciation rate decreases the growth rate of capital per worker.

If the depreciation rate is high, more capital wears out each year, leading to a low growth rate of capital per worker.

*Question:* Where do we see this in practice?

*Answer:* Before Castro came to power, Cuba had a high growth rate of capital per worker. Castro (and the U.S. embargo) closed the economy to foreign trade, and it was difficult to maintain equipment that needed foreign parts or foreign trained engineers. The depreciation rate increased, and the growth rate of capital per worker slowed. The Communist revolution in Cambodia emptied the cities and forced the population to the countryside. The capital was not maintained. The depreciation rate was greater than  $y/k$ , and the growth rate of capital per worker was negative. Socialism in Venezuela has caused oil production equipment to deteriorate and a once wealthy country is now poor.

*Part D:* A higher population growth rate decreases the growth rate of capital per worker.

If the labor force is growing faster than the capital stock, the growth rate of capital per worker is negative, and living standards do not improve.

*Question:* Where do we see this in practice?

*Answer:* In some sub-Saharan African countries, the population growth rate increased because of western medicine before the economy grew enough to handle the larger population. Real GDP per capita remained low. In post-World War II Japan, the population growth rate fell sharply, and a high regard for education led to repaid growth of capital.

*Question:* Countries with low fertility levels (Japan, Russia, Greece, Italy) face serious problems with mounting pension costs. Where is that considered in Barro's textbook? How does it fit into the Solow growth model?

*Answer:* Barro discusses this in the section of social security. In this part of the textbook, Barro assumes the growth rate of the labor force equals the population growth rate. In the countries with low fertility levels, the fertility rate declined drastically in the past generation. The labor force is declining, but the non-working population is not declining.

\*\* Exercise 4.4: Growth rate of capital per worker

- Countries ABC and XYZ are identical, except that ABC's savings rate is twice that of XYZ.
- Both countries follow the simple Solow growth model, with no changes in the technology level.
- The depreciation rate is 5% and the population growth rate is 1% in both countries.
- In 20X1, capital per worker is 10,000 and real GDP per worker is 2,000 in both countries.
- The growth rate of capital per worker in country XYZ is 0.5% per annum.

- A. What is the ratio of income to capital in these two countries?
- B. What is the savings rate in country XYZ?
- C. What is the growth rate of capital per worker in country ABC?

*Part A:*  $(y/k) = 2,000 / 10,000 = 20\%$  in both countries.

*Part B:* The growth rate of capital per worker in the Solow growth model is  $\partial k/k = s \times (y/k) - s \times \delta - n$

(See Barro, *Macroeconomics*, Chapter 4, page 51, equation 4.1.)

Use the values for XYZ to solve for the savings rate in country XYZ:

$$\text{XYZ: } 0.5\% = s \times 20\% - s \times 5\% - 1\% \Rightarrow 1.5\% = 15\% \times s \Rightarrow s = 10\%$$

*Part C:* Double the savings rate to get the growth rate of capital per worker in country ABC:

$$\text{ABC: } 20\% \times 20\% - 20\% \times 5\% - 1\% = 4\% - 1\% - 1\% = 2\%.$$

**\*\* Exercise 4.5: Solow growth model and changes in the savings rate**

An economy follows the simple Solow growth model.

- A. What is consumption per worker in terms of income per worker, capital per worker, the savings rate, and the depreciation rate?
- B. In the steady state, what is consumption per worker in terms of income per worker, capital per worker, the population growth rate, and the depreciation rate?
- C. In the steady state, what is the change in consumption per worker in terms of the change in capital per worker, the marginal product of capital, the population growth rate, and the depreciation rate?
- D. If the depreciation rate is 5%, the population growth rate is 1%, and the marginal product of capital is 16%, what is the change in steady state consumption per worker if a higher savings rate raises the steady state capital per worker by 100?

*Part A:* Real income,  $Y - \delta K$ , equals real net domestic product, which is consumption + savings:

$$Y - \delta K = C + s \times (Y - \delta K) \Rightarrow$$

$$C = (1 - s) \times (Y - \delta K) = Y - \delta K - s \times (Y - \delta K).$$

The equation per worker is  $c = y - \delta k - s \times (y - \delta k)$ .

*Part B:* Consumption equals real income minus savings, and savings in the steady state equals  $s \times (y^* - \delta k^*)$ , as shown in Part A:

$$c^* = y^* - \delta k^* - s \times (y^* - \delta k^*).$$

In the steady state, capital and income per worker do not grow; savings from current workers provides capital for new workers:

$$s \times (y^* - \delta k^*) = nk^*.$$

Substituting  $nk^*$  for  $s \times (y^* - \delta k^*)$  in the previous equations gives

$$c^* = y^* - \delta k^* - nk^*.$$

*Part C:* If the savings rate changes, steady state capital per worker changes by  $\Delta k^*$ , steady state income per worker changes by  $\Delta y^*$ , and steady state consumption per worker changes by  $\Delta c^*$ , giving

$$\Delta c^* = \Delta y^* - (\delta + n) \times \Delta k^*.$$

The change in income per worker is the marginal product of capital times the change in capital per worker:

$$\Delta y^* = MPK \times \Delta k^*.$$

The change in capital per worker is

$$\Delta c^* = (MPK - \delta - n) \times \Delta k^*.$$

*Part D:*  $\Delta c^* = (16\% - 5\% - 1\%) \times 100 = 10$ .

*Intuition:* Capital per worker increases 100, so income per worker increases  $16\% \times 100 = 16$ . Depreciation per worker increases  $5\% \times 100 = 5$ , so net income per worker increases  $(16\% - 5\%) \times 100 = 11$ . Capital of  $1\% \times 100 = 1$  is needed for new workers (who are 1% of existing workers), so capital per worker increases  $11 - 1 = 10$ .

**\*\* Exercise 4.6: Solow growth model and population growth rate**

An economy that follows the Solow growth model has a population growth rate of 3% per annum, with a 1% annual growth rate of capital per worker:  $\partial k/k = 1\%$ . If the population growth rate changes to 2% per annum

- A. What is the new annual growth rate of capital per worker?
- B. What is the new annual growth rate of capital per worker in the steady state?
- C. What is the new annual growth rate of capital in the steady state?

*Part A:* The formula for the growth rate of capital per worker is

$$\Delta k/k = sA \times f(k)/k - s\delta - n.$$

If  $n$  decreases from 3% to 2%,  $\Delta k/k$  increases from 1% to 2%.  $\Delta k/k$  increases when  $n$  decreases because less savings goes to provide additional workers with capital.

*Part B:* The growth rate of capital per worker in the steady state is always zero.

*Part C:* The growth rate of capital per worker in the steady state is the growth rate of capital per worker plus the population growth rate =  $0\% + 2\% = 2\%$ .

**\*\* Exercise 4.7: Solow growth model**

In the Solow growth model, income is a function of the savings rate  $s$ , the technology level  $A$ , the population growth rate  $n$ , and the capital depreciation rate  $\delta$ , and the labor force  $L$ . How does steady state capital per worker change as

- A. The savings rate  $s$  increases.
- B. The technology level  $A$  increases.
- C. The population growth rate  $n$  increases.
- D. The capital depreciation rate  $\delta$  increases.
- E. The current labor force  $L(0)$  increases.

*Part A:* A higher savings rate means more income is invested, so steady state capital per worker increases.

*Part B:* A higher technology level means each worker produces more income. If the savings rate does not change, more income is invested, so steady state capital per worker increases.

*Part C:* As the population growth rate increases, more capital is needed for new workers, so the steady state capital per worker decreases.

*Part D:* As the depreciation rate increases, capital per worker decreases.

*Part E:* If the current labor force increases, current capital per worker decreases, but steady state capital per worker doesn't change.

See Barro, *Macroeconomics*, Chapter 4, page 60, table 4.1, top of page.

**\*\* Exercise 4.8: Malthus updated**

The simple Solow growth model assumes a constant population growth rate. But modernization brings better health care and education to a country's population. In practice, what should we expect from Barro's revised Malthus model (assuming no change in the country's technology level over time) for each of the following?

- A. Life expectancies at birth
- B. Fertility rate
- C. Population growth rate
- D. Steady state capital per worker
- E. Growth rate of steady state capital per worker

*Part A:* Life expectancies at birth lengthen with greater development, from 30 to 40 years in primitive societies to 80 years in highly developed societies.

*Part B:* Fertility rates decline with modernization. People with more wealth and urban careers need not depend on children to help with cultivating land or to provide for them in old age. In addition, the opportunity to work at high income jobs creates a high opportunity cost to having children, especially for women, who bear the children and provide most of the child care. As families change from extended to nuclear, with grown children not living in the same household as parents, and as societies provide private and government pensions to older persons, the need for children diminishes.

*Part C:* The higher life expectancies overwhelm the lower fertility rate when societies first modernize, and most countries initially gain population. Once health care and education reach western levels, the lower fertility rate offsets the declining mortality, and populations contract (Europe, Russia, and non-Muslim Asian countries).

*Part D:* As the population growth rate declines, steady state capital per worker increases. The final term in the Solow growth model is " $-n$ ," or the negative of the population growth rate.

*Part E:* The growth rate of steady state capital per worker is zero if the country's technology level does not change over time. The change in the population growth rate has no effect.

See Barro, *Macroeconomics*, Chapter 4, page 62, "extending the model"

**\*\* Exercise 4.9: Solow growth model and Cobb-Douglas production function**

Assume South Korea follows a Cobb-Douglas production function with  $\alpha = 50\%$ . In 20XX, South Korea reunites with North Korea. The current labor input  $L(0)$  doubles, but no capital is added (North Korea has workers but little capital). South and North Korean workers do not differ (same population growth rate, savings rates, ability to use technology). Assume the technology level does not change.

- A. What is the effect on current capital per worker ( $k$ )?
- B. What is the effect on steady state capital per worker  $k^*$ ?
- C. What is the effect on the growth rate of capital per worker ( $k$ ) in the transition phase?
- D. What is the effect on the steady state capital stock  $K^*$ ?
- E. What is the effect on the growth rate of steady state capital per worker ( $k^*$ )?

*Part A:* Current capital per worker declines 50%, since the current labor input doubles.

*Part B:* Steady state capital per worker  $k^*$  does not change, since there are no differences in the attributes of South and North Koreans (same population growth rate, savings rates, ability to use technology, and so forth).

*Part C:* In the transition phase, the growth rate of capital per worker ( $k$ ) increases, since current capital per worker ( $k_0$ ) decreases and steady state capital per worker  $k^*$  doesn't change.

*Part D:* The steady state capital stock  $K^*$  doubles, since the labor force doubles.

*Part E:* The growth rate of steady state capital per worker ( $k^*$ ) is always zero in the steady state (assuming no changes in the technology level).

See Barro, *Macroeconomics*, Chapter 4

## \*\* Exercise 4.10: Solow Growth Model

In the Solow growth model, an economy's growth rate of real GDP depends on five variables:

- current capital per worker,  $k$
- the depreciation rate of its capital,  $\delta$
- the population growth rate,  $n$
- the savings rate,  $s$
- the technology level,  $A$

- A. How does current capital per worker  $k$  affect the economy's growth rate of real GDP (income)?
- B. How does the depreciation rate of capital  $\delta$  affect the growth rate of real GDP?
- C. How does the population growth rate  $n$  affect the growth rate of real GDP?
- D. How does the savings rate  $s$  affect the growth rate of real GDP?
- E. How does the technology level  $A$  affect the growth rate of real GDP?

*Part A: Current capital per worker:* Economies converge (revert) to their steady state capital and income per worker by absolute convergence or conditional convergence. Mean reversion (convergence) implies that as a random variable increases toward the mean, its rate of increase slows, and as the variable decreases toward the mean, its rate of decrease slows. A lower current capital per worker implies a faster growth of capital per worker during the transition phase.

*Part B: Depreciation*  $\delta$  is a reduction in capital. Higher depreciation means that more capital wears out and disappears from national income accounts.

*Parts C and D:* Savings provides for three items: replacement of capital that wears out (depreciation); capital for new workers (population growth); and higher capital per worker. A higher savings rate means that more remains after providing for depreciation and population growth, so capital per worker continues to grow. Conversely, a higher population growth rate means that more savings are needed to provide capital for new workers and less is available for growth in capital per worker.

Savings offsets the population growth rate and the depreciation rate, and it supplies funds for further growth in capital per worker. As capital per worker grows, the savings needed to support depreciation and population growth is larger. When capital is large enough that savings just offsets depreciation and population growth the economy is in its steady state.

Savings and population growth differ by country. They are (in part) culturally determined.

Some cultures place more stress on large families than others do, or they are more opposed to birth control, family planning, and abortion. Contrast the large families in the Middle East and Latin America with small families in Western Europe, Canada, and China. We do not know if economic growth in the Middle East and Latin America will reduce the population growth rate to European levels or if Islamic and Catholic cultures will continue with large families.

Some cultures have higher savings rates than others. Japan and other Asian countries have higher savings rates than North America and Western Europe.

- An economy with a high population growth rate and a low savings rate may be near its steady state capital per worker even at a low capital level.
- An economy with a low population growth rate and a high savings rate may be below its steady state capital per worker even at a high capital level.

*Part E:* Distinguish between capital and the technology level. A higher technology level raises the marginal product of capital and leads to more capital per worker. To remember this, think of two workers:

- Jacob lives in a shepherd culture, with no knowledge of industrial society ( $A$  is low). The only capital is knives, jars, and tents.
- Jacques lives in Paris and works in an information-age society ( $A$  is high). He uses cell-phones and laptop computers (high capital).

*Take heed:* Some exam problems specify the Solow growth model. Unless specified otherwise, the exam problems use the Solow growth model.

The Solow growth model has several forms: fixed values for  $A$ ,  $s$ , and  $n$ ; exogenous and endogenous changes in the parameters. The type of model should be clear from the context of the exam problem.

**\*\* Exercise 4.11: Solow growth model and Cobb-Douglas production function**

An economy has a Cobb-Douglas production function:  $Y = AK^\alpha L^{(1-\alpha)}$ .

$Y$  = income,  $K$  = capital,  $L$  = labor,  $y$  = income per worker, and  $k$  = capital per worker.

- A. What is the growth rate of income per worker in terms of the growth rates of income and labor?
- B. What is the growth rate of capital per worker in terms of the growth rates of capital and labor?
- C. What is the growth rate of income per worker in terms of  $\alpha$  and the growth rate of capital per worker if the technology level is constant?

*Part A:* The growth rate of income per worker is

$$\partial y/y = \partial(\ln(y)) = \partial(\ln(Y/L)) = \partial(\ln(Y) - \ln(L)) = \partial(\ln(Y)) - \partial(\ln(L)) = \partial Y/Y - \partial L/L$$

See Barro, macroeconomics, Chapter 3, Page 40, eq 3.6

*Part B:* The growth rate of income per worker is

$$\partial k/k = \partial(\ln(k)) = \partial(\ln(K/L)) = \partial(\ln(K) - \ln(L)) = \partial(\ln(K)) - \partial(\ln(L)) = \partial K/K - \partial L/L$$

See Barro, macroeconomics, Chapter 3, Page 40, eq 3.7

*Part C:* Divide both sides of the Cobb-Douglas production function by  $L$  (labor):

$$Y/L = y = (AK^\alpha L^{(1-\alpha)}) / L = A (K/L)^\alpha (L/L)^{(1-\alpha)} = Y = Ak^\alpha 1^{(1-\alpha)} = Ak^\alpha$$

$$\ln(y) = \ln(Ak^\alpha) = \ln(A) + \alpha \ln(k)$$

$$\partial [\ln(y)] \partial y/y \partial [\ln(A) + \alpha \ln(k)] = \alpha \times \partial [\ln(k)] = \alpha \times \partial k/k$$

See Barro, macroeconomics, Chapter 3, Page 40, eq 3.8

\*\* Exercise 4.12: Savings, investment, and capital

An economy follows a simple Solow growth model, with

$s$  = the savings rate

$\delta$  = the depreciation rate

$Y$  = real GDP

$K$  = capital

Inv = investment

- A. What is the relation between real savings and the savings rate?
- B. What is the relation between real savings and investment?
- C. What is the relation between real savings and the change in capital?
- D. What is the relation between real savings and the proportional change in capital?

*Part A:* Real savings is the savings rate times net income, which is gross income minus depreciation of capital (which is the depreciation rate times the capital stock).

$$\text{real savings} = s \times (Y - \delta K)$$

See Barro, *Macroeconomics*, Chapter 3, page 41, column 1, bottom.

*Part B:* Real savings = net investment, or

$$s \times (Y - \delta K) = \text{Inv} - \delta K$$

The left side of the equation is real savings from the previous equation. The right side says that net investment is gross investment minus depreciation of the capital stock.

See Barro, *Macroeconomics*, Chapter 3, page 41, equation 3.11

*Part C:* Net investment is the change in the capital stock, so

$$\Delta K = s \times (Y - \delta K)$$

See Barro, *Macroeconomics*, Chapter 3, page 41, equation 3.12

*Part D:* Divide the equation above by  $K$  to get

$$\Delta K/K = s \times Y/K - s \times \delta$$

See Barro, *Macroeconomics*, Chapter 3, page 41, equation 3.13