Macroeconomics Module 3: Cobb-Douglas production function practice problems

(The attached PDF file has better formatting.)

The final exam has three types of problems on economic growth

- Absolute and conditional convergence
- Solow growth model
- Cobb-Douglas production function

The homework assignments on economic growth are also tested on the final exam. One assignment discusses convergence (mean reversion) and stochasticity. Barro relates income inequality to mean reversion.

Know the attributes of the Cobb-Douglas production function. The final exam may test the steady state income and capital per worker, the elasticity of income with respect to capital or labor, the range of $\alpha$, and similar items. The formulas are in the appendix to Chapter 3.

For the transition path of the Solow growth model, know how the growth rate relates to steady state capital per worker, current capital per worker, savings, growth rate of technology, and population growth rate.

** Exercise 1.1: Cobb-Douglas Production Function

An economy has a Cobb-Douglas production function: $Y = AK^{\alpha} L^{(1-\alpha)}$.

*A* is the technology level, *K* is capital; *L* is labor; and *Y* is income.

Which of the following is true?

A. What is the definition of elasticity?
B. What is the elasticity of income with respect to the technology level?
C. What is the elasticity of income with respect to capital?
D. What is the elasticity of income with respect to labor?
E. What is the marginal product of capital?
F. What is the marginal product of labor?
G. What is the range of $\alpha$?
H. Does the Cobb-Douglas production function have constant returns to scale?
I. As $\alpha \rightarrow 0$, what is the steady state capital per worker?
J. As $\alpha \rightarrow 1$, what is the steady state capital per worker?

*Part A:* The elasticity of Y with respect to X is the percentage change in Y for a given percentage change in X, or $\partial Y/Y \div \partial X/X = \partial Y/\partial X \times (X/Y)$.

*Part B:* If the technology level increases 1% from A to 1.01 × A, Y becomes $1.01 \times AK^{\alpha} L^{(1-\alpha)}$. Y also increases 1%, so the elasticity of Y with respect to A is 100%, or 1.00. Mathematically, $\partial Y/\partial A = K^{\alpha} L^{(1-\alpha)}$, so the elasticity of Y with respect to A is $K^{\alpha} L^{(1-\alpha)} \times A / Y = 1$.

*Part C:* If capital increases 1% from K to 1.01 × K, Y becomes $1.01^{\alpha} \times AK^{\alpha} L^{(1-\alpha)}$. Y increases by a factor of $1.01^{\alpha}$. The relative increase of Y with respect to K depends on the size of the increase in K. The elasticity is the relative change in Y as the increase in K become infinitesimally small. Using the formula for the elasticity:

$\partial Y/\partial K = \alpha \times AK^{\alpha-1} L^{(1-\alpha)}$
$\partial Y/\partial K \times (K/Y) = \alpha \times AK^{\alpha-1} L^{(1-\alpha)} \times (K/Y) = \alpha$

*Part D:* The elasticity with respect to labor is like the elasticity with respect to capital, except the exponent is $(1 - \alpha)$, so the elasticity is $(1 - \alpha)$.

*Intuition:* If $Y = \alpha X$ + other terms, $\partial Y/\partial X = \alpha$.

Since $\partial Y/Y \div \partial X/X = \partial\, ln(Y) \div \partial\, ln(X)$, taking logarithms of the Cobb-Douglas production function gives

$$ln(Y) = ln(A) + \alpha\, ln(K) + (1 - \alpha)\, ln(L)$$

The elasticities are

- $\partial\, ln(Y) / \partial\, ln(A) = 1$
- $\partial\, ln(Y) / \partial\, ln(K) = \alpha$
- $\partial\, ln(Y) / \partial\, ln(L) = (1 - \alpha)$

*Part E:* For the Cobb-Douglas production function, the marginal product of capital is

$MPK = \partial Y / \partial K$
$= \alpha\, AK^{\alpha-1} L^{1-\alpha}$
$= \alpha\, AK^{\alpha} K^{-1} L^{1-\alpha}$
$= \alpha\, AK^{\alpha} L^{1-\alpha} \times (1/K)$
$= \alpha \times (Y/ K)$

The marginal product of capital decreases as capital increases.

*Note:* The elasticity of income with respect to capital is $\partial Y / \partial K \times (K / Y)$ = MPK × K / Y

= $\alpha \times (Y/ K) \times K / Y = \alpha$ (see equation 3.20 on page 67)

*Part F:* The marginal product of labor is similar to the marginal product of capital except that the exponent is $(1 - \alpha)$ instead of $\alpha$. The marginal product of labor is $(1 - \alpha) \times (Y / L)$.

The marginal product of capital is the effect on Y from a change in K, holding A and L fixed, and the marginal product of labor is the effect on Y from a change in L, holding A and K fixed.

The elasticity of income with respect to capital is the percentage change in income from a percentage change in capital. If $\alpha$ = 40%, a 5% increase in capital raises income 5% × 40% = 2%, and a 5% rise in labor raises income 5% × (1 – 40%) = 3%.

Elasticity is like a derivative, except:

- The derivative of income with respect to capital is $\partial Y / \partial K$.
- The elasticity of income with respect to capital is $(\partial Y/Y) / (\partial K/K) = \partial Y / \partial K \times (K/Y)$.

*Part G:* The range of $\alpha$ is [0, 1].

- More capital increases income, so $\alpha > 0$. If $\alpha < 0$, adding capital reduces income.
- More labor increases income, so $\alpha < 1$. If $\alpha > 1$, increasing labor decreases income.

*Part H:* Raising both capital and labor Z% raises income Z%. ⇛ The Cobb-Douglas production function has constant returns to scale.

*Part I:* If $\alpha \approx 0$, capital adds nothing to income, but labor adds much to income. Capital costs money (the real rental price), so the optimal amount of capital is zero. The steady state capital per worker is zero.

In practice, $\alpha$ can not be zero. Capital includes simple tools and human knowledge (human capital). If workers have no knowledge and no tools, they can not produce anything.

*Part J:* If $\alpha \approx 1$, labor adds almost to income, but capital adds much to income. Labor costs money (the real wage rate), so the optimal amount of labor is zero. The steady state capital per worker $\rightarrow \infty$.

In practice, $\alpha$ can not be 1. With no labor, capital is useless.

** Exercise 1.2: Cobb-Douglas Production Function

An economy has a Cobb-Douglas production function: $Y = AK^{\alpha} L^{(1-\alpha)}$, with $\alpha$ = ⅔ (66.7%).

A. If the technology level increases 6%, what is the increase in income?
B. If capital increases 6%, what is the increase in income?
C. If labor increases 6%, what is the increase in income?
D. If the technology level, capital, and labor all increases 6%, what is the increase in income?

If labor increases 2%, capital increases 5%, and the technology level increases 4%, what is the increase in real GDP?

*Part A:* The elasticity of income with respect to the technology level is 100%, so income increases 6%.

*Part B:* The elasticity of income with respect to capital is $\alpha$, so income increases 6% × ⅔ = 4%.

*Part C:* The elasticity of income with respect to labor is $(1 - \alpha)$, so income increases 6% × (1 – ⅔) = 2%.

*Part D:* We add elasticities, so income increases 6% + 4% + 2% = 12%.

** Exercise 1.3: Cobb-Douglas Production Function and Solow growth model

- An economy has a Cobb-Douglas production function: $Y = AK^{\alpha} L^{(1-\alpha)}$, with $\alpha$ = ¾ (75%).
- The economy follows the simple Solow growth model, with no growth in the technology level.

- The depreciation rate is 5%, the population growth rate is 2%, the savings rate is 15%, and the technology level is 1.50.
- The annual growth rate of capital per worker is 4%.

What is the annual growth rate of real GDP per worker (y*)?

Solution 1.3: $\Delta y/y = \alpha \times \Delta k/k \Rightarrow 75\% \times 4\% = 3\%$.

See Barro macroeconomics Chapter 4, page 61, equation 4.8

$\alpha$ is the capital share coefficient. If capital per worker increases 1%, real GDP per worker increases $\alpha \times 1\%$.

*Question:* What about the other parameters? If the savings rate is higher, won't the growth rate of real GDP per worker be higher? And if the population growth rate is higher, won't the growth rate of real GDP per worker be lower?

*Answer:* The other parameters affects the growth rate of capital per worker in the same manner as the growth rate of real GDP per worker. If the savings rate is higher or if the population growth rate is lower, the growth rate of capital per worker and real GDP per worker are higher by the same degree.

** Exercise 1.4: Cobb-Douglas production function

An economy has a Cobb-Douglas production function.

- In 20X1, the technology level A is 100, capital K = 200, labor L = 400, and income Y = 10,000.
- In 20X2, the technology level A is 101, capital K = 212, labor L = 412, and income Y = 10,500.
- In 20X3, the technology level A is 103, capital K = 231, labor L = 424.

A. What are the percentage changes for A, K, L, and Y?
B. From the figures for 20X1 and 20X2, derive the $\alpha$ parameter of the Cobb-Douglas production function.
C. From the figures for 20X2 and 20X3, derive income (Y) in 20X3.

*Part A:* Using approximations to one decimal place, the percentage changes are

- From 20X1 to 20X2, the percentage changes are 1% for A, 6% for K, 3% for L, and 5% for L.
- From 20X1 to 20X2, the percentage changes are 2% for A, 9% for K, and 2.9% for L.

*Part B:* The change in the capital stock is not the same as the change in the labor supply, so the change i Y depends on the $\alpha$ parameter of the Cobb-Douglas production function.

Using the change from 20X1 to 20X2:

$\partial Y/Y = \partial A/A + \alpha \times \partial K/K + (1 - \alpha) \times \partial L/L = 1\% + \alpha \times 6\% + (1 - \alpha) \times 3\% = 5\% \Rightarrow \alpha = ⅓.$

*Part C:* Using the change from 20X2 to 20X3:

$\partial Y/Y = \partial A/A + \alpha \times \partial K/K + (1 - \alpha) \times \partial L/L = 2\% + \alpha \times 9\% + (1 - \alpha) \times 3\% = 6.9\% \Rightarrow L = 10{,}500 \times 1.069 = 11{,}225$

** Question 1.5: Production function

$Y = A \times F(K,L)$, where F is a Cobb-Douglas production function: $Y = AK^{\alpha} L^{(1-\alpha)}$, with $\alpha = 40\%$.

If A, K, and L all increase 2%, what is the change in Y?

Answer 1.5: $1.02 \times 1.02 - 1 = 4.04\%$

The Cobb-Douglas production function has constant returns to scale: if capital and labor increase P%, income increases P%.

The technology level is a multiplicative factor: if A increases P%, income increases P%.

The change in income in this exercise is $(1.20 \times 1.20) - 1 = 44\%$.

See Barro *Macroeconomics* Chapter 3 Introduction to Economic Growth, page 36

** Question 1.6: Production function

$Y = A \times F(K,L)$, where Y = income, A = technology level, K = capital, and L = labor.

Which of the following is a possible production function F?

A. $F = K \times L$
B. $F = ln(K \times L) - 100$
C. $F = K^{0.5} \times L^{0.5}$
D. $F = (K^{0.5} \times L^{0.5})^2$
E. $F = K^{0.5} + L^{0.5}$

Answer 1.6: C

*Statement A:* The production function K × L does not have decreasing marginal utility for either capital or labor, since $\partial(K \times L)/\partial K = L$ and $\partial(K \times L)/\partial L = K$

*Statement B:* The production function $ln(K \times L) - 100$ does not pass through the origin. If either K is zero or L is zero, income is zero.

*Statement D:* The production function $(K^{0.5} \times L^{0.5})^2$ does not have decreasing marginal utility for capital or labor.

$\partial[(K^{0.5} \times L^{0.5})^2]/\partial K = 2\,(K^{0.5} \times L^{0.5}) \times 0.5K^{-0.5} \times L^{0.5} = LK$

*Take heed:* Both labor and capital have decreasing marginal utility: the marginal product of labor decreases as labor increases and the marginal product of capital decreases as capital increases. The technology level may or may not have decreasing marginal utility; Barro presents several models.

*Statement E:* The production function $K^{0.5} + L^{0.5}$ does not equal zero when K or L is zero. To be realistic, a production function should be zero when labor or capital is zero.

*Question:* With no workers, production is zero. But with no capital (such as no tractors), workers can still do some things by hand, so shouldn't production by more than zero?

*Answer:* Even the simplest tools are capital, such as an ox to plough the ground or a knife to prune trees. For industrial work, nothing is produced without machines and equipment. In later modules, human capital is included with capital. Workers' education is defined by some economists as part of capital. Workers with no tools and no education produce nothing.

See Barro *Macroeconomics* Chapter 3 Introduction to Economic Growth, page 36

Exercise 1.7: Cobb-Douglas production function

An economy has a Cobb-Douglas production function: $Y = AK^{\alpha} L^{(1-\alpha)}$, with $\alpha$ = 66.7%.

A. What is the elasticity of income with respect to capital?
B. What is the elasticity of income with respect to labor?
C. What is the elasticity of income with respect to the technology level?
D. If labor increases 2%, capital increases 5%, and the technology level increases 4%, what is the increase in real GDP?

*Part A:* The elasticity of income with respect to capital is $\Delta Y/Y \div \Delta K/K$, or the percentage change in income divided by the percentage change in capital. Barro calls this the capital share coefficient of F(K,L). If capital increases 1%, income increases $1.01^{0.667} - 1 = 0.666\%$, so the elasticity is 66.7%. (The 0.1% difference is a rounding error; for a minute increase in capital, the elasticity is exact.)

*Part B:* If labor increases 1%, income increases 1 – 66.7% = 33.3%, so the elasticity is 33.3%.

*Part C:* If the technology level increases 1%, income increases 1%, so the elasticity is one.

*Part D:* Barro's equation 3.4 is $\Delta Y/Y = \Delta A/A + \alpha \times \Delta K/K + (1 - \alpha) \times \Delta L/L$.

Real GDP increases 4% + 5% × 66.7% + 2% × 33.3% = 8.00%.