Corporate Finance, Module 6: “Risk, Return, and Cost of Capital”

Practice Problems

(The attached PDF file has better formatting.)

Exercise 6.1: Minimum Variance Portfolio

Port(A,B) is a combination of two stocks, A and B, with standard deviations σ_A and σ_B. $\rho_{A,B}$ = correlation (A,B) = 0.

A. If the two stocks are equally weighted, what is the variance of Port(A,B)?
B. If the weight for Stock A is ω_A, what is the variance of Port(A,B)?
C. If Port(A,B) is the minimum variance portfolio, what is the weight for stock A?

Solution 6.1:

Part A: If the stocks are equally weighted and their correlation is zero, the variance of the portfolio is ¼ σ_{A2} + ¼ σ_{B2}.

Part B: If the weight for Stock A is ω_A, the variance of the portfolio Port(A,B) is

$$\omega_{A2}\ \sigma_{A2} + 2\ \omega_A\ (1 - \omega_A)\ \rho_{A,B}\ \sigma_A\ \sigma_B + (1 - \omega_A)^2\ \sigma_{B2}$$

If $\rho_{A,B} = 0$, the variance of the portfolio is $\omega_{A2}\ \sigma_A^2 + (1 - \omega_A)^2\ \sigma_{B2}$.

Part C: We minimize the variance by setting the partial derivative with respect to ω_A equal to zero:

$$2\ \omega_A\ \sigma_{A2} - 2\ (1 - \omega_A)\ \sigma_{B2} = 0$$
$$\omega_A = \sigma_B^2 / (\sigma_{A2} + \sigma_{B2})$$

This is the Bayesian-Bühlmann credibility, which is a minimum variance linear estimator. To minimize the variance of the portfolio, we weight the two stocks by the complements of their relative variances. If one stock has 80% of the combined variance, it gets 20% of the weight.

{The final exam does not test the mathematics of minimum variance portfolios. We show this problem since similar problems may appear on the CAS transition exam.}

Exercise 6.2: Minimum Variance Portfolio

Port(A,B) is a combination of two stocks, A and B, with standard deviations σ_A and σ_B. $\rho_{A,B}$ = correlation (A,B) $\neq 0$.

A. If the two stocks are equally weighted, what is the variance of Port(A,B)?
B. If the weight for Stock A is ω_A, what is the variance of Port(A,B)?
C. If Port(A,B) is the minimum variance portfolio, what is the weight for stock A?

Solution 6.2:

Part A: If the weight for Stock A is ω_A, the variance of the portfolio Port(A,B) is

$$\omega_{A2}\ \sigma_{A2} + 2\ \omega_A\ (1 - \omega_A)\ \rho_{A,B}\ \sigma_A\ \sigma_B + (1 - \omega_A)^2\ \sigma_{B2}$$

Part B: If the stocks are equally weighted, the variance of the portfolio is

$$\tfrac{1}{4}\ \sigma_{A2} + \tfrac{1}{4}\ \sigma_{B2} + \tfrac{1}{2}\ \rho_{A,B}\ \sigma_A\ \sigma_B.$$

Part C: We minimize the variance by setting the partial derivative with respect to ω_A equal to zero:

$$2\ \omega_A\ \sigma_{A2} + 2\ (1 - 2\omega_A)\ \rho_{A,B}\ \sigma_A\ \sigma_B - 2\ (1 - \omega_A)\ \sigma_{B2} = 0$$
$$\omega_A = (\sigma_{B2} - \rho_{A,B}\ \sigma_A\ \sigma_B)\ /\ (\sigma_{A2} + \sigma_{B2} - 2\ \rho_{A,B}\ \sigma_A\ \sigma_B)$$

Question: Are we increasing or decreasing the weights?

Answer: Suppose Stock B has four times the variance of Stock A. If the stocks were independent, we would give Stock A 80% of the weight and Stock B 20% of the weight.

If the correlation between the stocks is positive, we *add* the covariance term to the variance of the portfolio. The covariance term has the coefficient 2 ω_A $(1 - \omega_A)$. To minimize this term, we want to make ω_A close to zero or one. If the weight for Stock A is less than that for Stock B, we reduce its weight further; if it is more, we increase its weight further.

If the correlation between the stocks is negative, we *subtract* the covariance term from the variance of the portfolio. The covariance term has the coefficient 2 ω_A $(1 - \omega_A)$. To minimize this term, we want to make ω_A close to one half. If the weight for Stock A is less than that for Stock B, we increase its weight; if it is more, we decrease its weight.

Question 6.3: Price Weighted vs Value Weighted Returns

(Adapted from question 5 of the Spring 1997 Course 2 examination)

Which of the following is true?

A. The return on a portfolio is a *price weighted* average of the individual stocks' returns.
B. The variance of a portfolio is a *price weighted* average of the individual stocks' variances.
C. The standard deviation of returns on a portfolio of two uncorrelated stocks is a *price weighted* average of the standard deviation of the individual stocks.
D. The variance of a well-diversified portfolio reflects mainly the covariance of the stocks in the portfolio.
E. None of A, B, C, or D is true.

Answer 6.3: D

The statements are statistical principles, not peculiar to investment analysis.

Statement A: The expectation of the sum of random variables is the sum of the expectations:

$$E(\alpha + ß) = E(\alpha) + E(ß)$$

The return is the expected income divided by the initial value. Let the initial values of the random variables be A (for α) and B (for ß). The expected return on $(\alpha + ß)$ is

$$E(\alpha + ß) / (A + B) = [E(\alpha) / A] \times [A / (A + B)] + [E(ß) / B] \times [B / (A + B)]$$

This is the *value* weighted average of the expected returns on stocks A and B. The *price* weighted average considers the stock prices, not the market value of the stocks.

Illustration: Suppose 100 shares of Stock A sell for \$80 apiece and return 7%, and 1,000 shares of stock B sell for \$60 apiece and return 5%. The *price weighted average* is

$$(\$80 \times 7\% + \$60 \times 5\%) / \$140 = 6.14\%$$

The *value weighted average* is

$$(100 \times \$80 \times 7\% + 1{,}000 \times \$60 \times 5\%) /(100 \times \$80 + 1{,}000 \times \$60) = 5.24\%$$

Statements B and C: The standard deviation of the stock combination is a weighted average of the standard deviations of the two stocks only if they are perfectly correlated. If the standard deviations of stocks A and B are σ_A and σ_B, their weights in the portfolio are ω_A and ω_B (with $\omega_B = 1 - \omega_A$) and their correlation is $\rho_{A,B}$, the variance of the portfolio $\omega_A \times A + \omega_B \times B$ is

$$(\omega_{A2}\sigma_{A2} + 2\rho\omega_A\omega_B\sigma_A\sigma_B + \omega_{B2}\sigma_{B2})$$

and the standard deviation of the portfolio $\omega_A \times A + \omega_B \times B$ is

$$(\omega_{A2}\sigma_{A2} + 2\rho\omega_A\omega_B\sigma_A\sigma_B + \omega_B^2\sigma_B^2)^{½}$$

- If $\rho < 1$, $(\omega_{A2}\sigma_{A2} + 2\rho\omega_A\omega_B\sigma_A\sigma_B + \omega_{B2}\sigma_{B2})^{½} < (\omega_A\sigma_A + \omega_B\sigma_B)$.
- If $\rho = 1$, $(\omega_{A2}\sigma_{A2} + 2\rho\omega_A\omega_B\sigma_A\sigma_B + \omega_{B2}\sigma_{B2})^{½} = (\omega_A\sigma_A + \omega_B\sigma_B)$.

Statement D: The variance of an equally weighted portfolio of N stocks, each of which has a standard deviation of σ and each pair of which has a correlation of ρ, is the sum of the variances and covariances among all the stocks. The variance/covariance matrix has N^2 entries: N are variances (along the major diagonal) and the remaining N × N-1 are covariances [= 2 × ½N(N-2)]. Each cell has a weight of $1/N^2$. The sum of the cells is σ^2 / N + ρ σ^2 (N-1)/N

$$\sum \frac{1}{N^2} N\sigma^2 + \sum N(N-1)\frac{1}{N^2}\rho\sigma^2$$

The limit as N → ∞ of this expression is $\rho\ \sigma^2$.

Exercise 6.4: Market Portfolio

(Adapted from question 24 of the Spring 1997 actuarial examination)

The market portfolio has the following range of returns. The covariance between the returns on stock A and the market returns is 8%, and the risk-free rate of return is 5%. Use the CAPM to answer the questions below.

Probability	*Market Return*
10%	–50%
30%	–10%
30%	20%
20%	35%
10%	70%

A. What is the variance of the market portfolio?
B. What is the β of Stock A?
C. What is the expected return on stock A?

Solution 6.4:

Part A: We determine the variance of the overall market return:

Probability	*Return*	*Deviance*	*Deviance*^{2}
10%	-50%	-62%	38.440%
30%	-10%	-22%	4.840%
30%	20%	8%	0.640%
20%	35%	23%	5.290%
10%	70%	58%	33.640%
Average	12%	0%	9.910%

(This is a population variance, not a sample variance.)

Part B: The beta for stock A is the covariance between the returns on the market and those on stock A divided by the variance of the market returns, or 0.08 / 0.0991 = 80.73%.

Part C: The expected return on the market portfolio is 12%, so the expected return on stock A is 5% + 80.73% × (12% – 5%) = 10.65%.

{The corporate finance final exam does not test the mathematics of regression analysis. We show this problem since similar problems may appear on the CAS transition exam.}

Exercise 6.5: Expected Rate of Return

(Adapted from question 6 of the Fall 1997 actuarial examination)

Assume that average historical returns are as follows:

Investment Type	Average Annual Nominal Rate of Return	Average Annual Real Rate of Return
Treasury Bills	5.5%	1.0%
Corporate Bonds	7.5%	3.0%
Common Stocks	11.5%	7.0%

If next year's Treasury bill interest rate is 7%, and there is a stable risk premium on each investment type, what is the expected nominal rate of return for common stocks next year?

Answer 6.5: 13%

The market risk premium is the overall market return minus the risk-free interest rate, or 11.5% – 5.5% = 6.0%. If the risk-free rate is 7% next year, the expected return on the market is 7% + 6% = 13%.

Question 6.6: Risks

(Adapted from question 7 of the Fall 1997 actuarial examination)

Which of the following is true?

A. Because there is no risk of default, an investor can lock in a *real* rate of return by purchasing a Treasury security.
B. If the average covariance between securities were zero, one could eliminate all risk by holding a sufficient number of securities.
C. If two bonds have the same default risk, they have the same expected return.
D. The greatest payoff to diversification comes when two stocks are uncorrelated.
E. None of A, B, C, or D is true.

Answer 6.6: B

Statement A: The real return is the nominal return minus the inflation rate (or one plus the nominal return divided by one plus the inflation rate). If inflation changes unexpectedly, the investor's *real* return may change, even without default risk.

Statement B: The variance of the portfolio approaches the average covariance of the pairs of stock in the portfolio. As the average covariance approaches zero, the variance of the portfolio approaches zero.

Statement C: The return depends on maturity (and other items besides default risk). With an upward sloping term structure of interest rates, longer maturity bonds have higher yields than shorter maturity bonds, even if they have the same default risk.

Statement D: If the standard deviations of stocks A and B are σ_A and σ_B, their weights in the portfolio are ω_A and ω_B and their correlation is $\rho_{A,B}$, the standard deviation of the portfolio $\omega_A \times A + \omega_B \times B$ is $(\omega_{A2}\sigma_A^2 + 2\rho\omega_A\omega_B\sigma_A\sigma_B + \omega_{B2}\sigma_B^2)^{½}$. To minimize the standard deviation of the portfolio, we choose $\rho = -(\omega_{A2}\sigma_A^2 + \omega_{B2}\sigma_{B2}) / 2\omega_A\omega_B\sigma_A\sigma_B$, or as close to this value as we can (since ρ cannot be less than -1).

Exercise 6.7: Diversification

(Adapted from question 26 of the Fall 1996 actuarial examination)

The CEO of a publicly traded general liability insurance company wants to buy another publicly traded insurer. For each of Parts A and B, explain whether the proposed acquisition is in the best interests of shareholders.

A. The target company writes personal auto insurance; the purpose of the acquisition is to diversify the insurance writings.
B. The target company writes commercial auto insurance; the purpose of the acquisition is to reduce underwriting and acquisition expenses.

Solution 6.7:

Part A: Investors can diversify more efficiently than the company itself can. If the investors want to diversify, they can buy stock in a personal lines insurer. Most diversification by companies helps managers, who fear taking risks that might endanger their jobs, not the stockholders, who gain from aggressive firms. Management should focus on maximizing the company's expected return and let shareholders manage their risk.

Part B: Reducing expenses by economies of scale in the acquisition increases the expected return and increases the expected gain for shareholders. Acquisitions that create operating gains help both managers and shareholders.

In certain cases, diversification by companies helps shareholders. Two examples are

(1) Diversification that reduces the risk of insolvency helps shareholders if the costs of bankruptcy are high. Brealey and Myers discuss this topic in chapters 18 and 19, regarding optimal capital structure. It is rare for the costs of bankruptcy to be that high that it pays for the company to diversify.
(2) Acquisition of a firm with large tax losses by a profitable company can reduce federal income taxes. This has nothing to do with diversification; it is tax strategy.

Question: Won't diversification lower the capitalization rate for the stock?

Answer: The capitalization rate depends on the firm's beta. The beta of the merged firms is the weighted average of the betas of the two original firms. Diversification does not affect the beta other than by the weighted average.

Question 6.8: Risk Minimization

(Adapted from question 2 of the Spring 1998 actuarial examination)

Suppose that stocks Y and Z have a perfect negative correlation, and the variance of returns is 225% for stock Y and 400% for stock Z.

A. What is σ_Y, the standard deviation of stock Y? (Write the variance as a decimal, not a percentage, and take the square root.)
B. What is σ_Z, the standard deviation of stock Z?
C. What is $\rho_{Y,Z}$, the correlation of stocks Y and Z?
D. What is $covar_{Y,Z}$, the covariance of stocks Y and Z?
E. If ω_Y is the percentage of funds invested in stock Y, what is the variance and standard deviation of the portfolio?
F. If we invest 50% of the funds in each stock, what is the variance and standard deviation of the portfolio?
G. What value of ω_Y minimizes the variance and standard deviation of the portfolio? Solve this by setting the partial derivative of the variance with respect to ω_Y equal to zero.
H. What is the standard deviation of this minimum variance portfolio?

Solution 6.8:

Part A: The standard deviation is the square root of the variance, so $\sigma_Y = 2.25^{½} = 1.50$.

Part B: The standard deviation is the square root of the variance, so $\sigma_Z = 4.00^{½} = 2.00$.

Part C: The correlation $\rho_{Y,Z}$ is –1 (the meaning of perfect negative correlation).

Part D: The covariance $covar_{Y,Z}$ is –1 × 1.50 × 2.00 = –3.00.

Part E: The variance of the portfolio is [$\omega_{Y2}\sigma_{Y2} - 2\omega_Y(1-\omega_Y)\sigma_Y\sigma_Z + (1-\omega_Y)^2\sigma_{Z2}$] and the standard deviation is the square root of this.

Part F: If ω_Y = 50%, the variance is 0.25 × 2.25 – 2 × 0.5 × 0.5 × 1.5 × 2.00 + 0.25 × 4.00 = 0.063, and the standard deviation is $0.063^{½} = 0.251$.

Part G: To minimize the variance of the portfolio, we set its partial derivative with respect to ω_Y to zero: ∂ [$\omega_{Y2}\sigma_{Y2} - 2\omega_Y(1-\omega_Y)\sigma_Y\sigma_Z + (1-\omega_Y)^2\sigma_Z^2$] / $\partial\omega_Y = 0 \Rrightarrow$

$$2\omega_Y \times 2.25 - 2 \times 1.5 \times 2 \times (1 - 2\omega_Y) - 2 \times (1 - \omega_Y) \times 4 = 0$$
$$24.5\ \omega_Y = 14$$
$$\omega_Y = 14 / 24.5 = 57.14\%$$

57.14% of the funds should be invested in Stock Y and 42.86% in Stock Z.

Part H: The standard deviation of the portfolio is

$$57.14\%^2 \times 2.25 - 2 \times 57.14\% \times (1 - 57.14\%) \times 1.5 \times 2 + (1 - 57.14\%)^2 \times 4 = 0.00\%.$$

Question 6.9: Investment Risk

(Adapted from question 3 of the Spring 1998 actuarial examination)

Which of the following is true?

A. Treasury securities have no default risk and no purchasing power risk.
B. The coupon yield on a firm’s bonds cannot be greater than the expected return on its stock.
C. Investors value companies that diversify to reduce risk.
D. According to the CAPM, the market rate of return = the risk-free interest rate + a premium for risk.
E. None of A, B, C, or D is true.

Answer 6.9: D

Statement A: (false) Treasury securities have inflation risk (purchasing power risk), even if they have no default risk.

Statement B: (false) Coupons yields can be any amount; if the coupon yield is high, the market value of the bond is high. A firm which issued high coupon bonds when it started up (and was a high risk form) but which became less risky as it matured may have an expected return on its common stock that is lower than the coupon yield on its bonds.

Illustration: Suppose the risk-free rate is 6% and the market risk premium is 8%. A high tech stock company begins operations in 20X2, and its stock trades at a β of 2, for a 6% + 2 × 8% = 22% expected return. In 20X3, the firm issues 30 year bonds at a interest rate of 18%, or 12% above the risk-free rate (and probably about 900 basis points to 1,000 basis points above the 30 year Treasury rate). By 20X8, the firm is stable and profitable, and its β drops to 1.25, for an expected stock return on 6% + 1.25 × 8% = 16.00%.

Statement C: (false) Shareholders can diversify more easily and efficiently than firms can. Investors value firms that focus on their strengths and do not try to replicate matters that shareholders can do better.

Statement D: (true) This is the CAPM’s expression of the market rate of return, which Brealey and Myers use through most of their text.

Exercise 6.10: Equal-Weighted Portfolio

(Adapted from question 24 of the Fall 1998 actuarial examination)

What is the variance of a portfolio consisting of equal weights of stocks A and B?

$$\rho_{AB} = 0.6,\ \sigma_{A2} = 0.7,\ \sigma_{B2} = 0.6$$

Solution 6.10: The variance of the equal weighted portfolio of stocks A and B is

$$50\%^2 \times 0.7 + 0.6 \times 2 \times 50\% \times 50\% \times 0.7^{½} \times 0.6^{½} + 50\%^2 \times 0.6 = 51.94\%.$$

Exercise 6.11: Stock Statistics

(Adapted from question 25 of the Fall 1998 Course 2 examination)

A portfolio consists of equal portions (equal holdings) of N stocks, each of which has a standard deviation of 25%, and the correlation coefficient between each pair of stocks is 30%. The market portfolio has a standard deviation of 20%.

A. If N = 2, what is the standard deviation of the portfolio of the two stocks?
B. What is the standard deviation of the returns in terms of N?
C. As N approaches infinity, what is the standard deviation of the portfolio?
D. If the portfolio is fully diversified, what is its beta with N = infinity?
E. What is the covariance between this portfolio and the market?

Solution 6.11:

Part A: The variance of the portfolio is $\frac{1}{4} \times 25\%^2 + 2 \times 30\% \times \frac{1}{2} \times \frac{1}{2} \times 25\% \times 25\% + \frac{1}{4} \times 25\%^2 = 4.0625\%$, and the standard deviation is $4.0625\%^{\frac{1}{2}} = 20.156\%$.

Part B: If the standard deviations of stocks A and B are σ_A and σ_B, their weights in the portfolio are ω_A and ω_B, where $\omega_B = 1 - \omega_A$, and their correlation is $\rho_{A,B}$, the standard deviation of the portfolio $\omega_A \times A + \omega_B \times B$ is $(\omega_{A2}\ \sigma_{A2} + 2\ \rho_{A,B}\ \omega_A\ \omega_B\ \sigma_A\ \sigma_B + \omega_{B2}\ \sigma_B{}^2)^{\frac{1}{2}}$. In this exercise, all the correlations are 0.3 and all the standard deviations are 25%. The standard deviation of the portfolio is

$$(N \times 1/N^2 \times 25\%^2 + N \times (N-1) \times 0.3 \times 1/N \times 1/N \times 25\% \times 25\%)^{\frac{1}{2}}$$
$$= 25\% \times (1/N + \{ (N-1) / N \} \times 0.3)^{\frac{1}{2}}$$

Part C: As $N \to \infty$, the standard deviation above $\to 25\% \times 0.3^{\frac{1}{2}} = 0.1369 = 13.69\%$.

Part D: Beta is the covariance of the portfolio with the overall market divided by the variance of the market, or

$$ß = \text{covar}(r_p, r_m) / \text{var}(r_m) = \rho(r_p, r_m) \times \sigma_p \times \sigma_m / \sigma^{2m} = \rho(r_p, r_m) \times \sigma_p / \sigma_m$$

If the portfolio is fully diversified, its correlation with the overall market return is one, so $ß_p = 13.69\% / 20\% = 0.685$.

Part E: If the correlation is one, the covariance is $1 \times 20\% \times 13.69\% = 2.74\% = 0.0274$.