Corporate Finance Mod 23: Options, Black-Scholes, Practice Problems

** Exercise 23.1: Black-Scholes

A stock trades at price S. The risk-free interest rate is *r* and the stock price volatility is σ.

A put option on this stock has an exercise price of X and a time to maturity of *t*.

N is the cumulative distribution function of the standard normal distribution.

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma(\sqrt{T-t})}$$

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma(\sqrt{T-t})} = d_1 - \sigma\sqrt{(T-t)}$$

A. What is the value of the call option?
B. What is the value of the put option?

Part A: The value of the call option is S × $N(d_1) - Xe^{-rt} \times N(d_2)$.

Part B: The value of the put option is $Xe^{-rt} \times N(-d_2) - S \times N(-d_1)$.

Question: What is the relation of $N(d_1)$ to $N(-d_1)$?

Answer: A cumulative density function ranges from 0 to 1. The normal distribution is symmetric about 0, so

N(0) = 0.5 and $N(-d_1) = 1 - N(d_1)$.

** Exercise 23.2: Input Parameters to the Black-Scholes formula

A. What are the five input parameters to the Black-Scholes formula for the value of a European call or put option on a non-dividend paying stock?
B. Which input parameters are stated in the option contract?
C. Which input parameters are known from market values?
D. Which input parameters must be estimated by formulas?

Part A: Five input parameters affect the value of a European call or put option on a non-dividend paying stock: stock price, strike price, volatility, risk-free interest rate, and term to maturity.

Part B: The strike price and the maturity date are stated in the option contract.

Part C: The stock price and the risk-free rate are known from market values. The risk-free rate may not be known exactly, but Treasury securities, LIBOR rates, and swap rates are good proxies.

Part D: The stock price volatility must be estimated by formulas. (The formulas are not in the textbook.)

The *strike price* (exercise price) and the *maturity date* (expiration date) of the option are chosen by the buyer and seller. They are stated in the option contract. Most option contracts conform to standards: the exercise price is in multiples of $5: an option may have an exercise price of $65 but not $66 or $67. Most traded options have times to maturity of 3 months to a year, but other times are sometimes used. Most option contracts conform to rules about the expiration date within each month, but other dates may also be used.

The *stock price* is a market value: the trading price of the stock.

The *risk-free interest rate* is the rate on Treasury securities or LIBOR securities with the same maturity as the option or the swap rate for the maturity. The risk-free rate in not known exactly, but the uncertainty is small.

The stock price *volatility* is estimated with great uncertainty. We can estimate the volatility of past stock price movements, but since these movements have high volatility, our estimate may be over-stated or under-stated. In addition, the stock price volatility changes as the environment changes; even if we knew the past volatility with certainty, we do not know the future volatility with certainty. The uncertainty in measures of stock price's volatility are so great that financial economists do not use statistical estimates of a stock price's variance to derive the volatility. They use *implied volatility*: that is, they use the prices of traded options and the Black-Scholes formula to back out the volatility.

** Exercise 23.3: Stock price distribution

Suppose the current stock price is S, the expected return on the stock is R% per annum, the strike price on a one year option is X, and the risk-free rate is r_f per annum.

A. What does it mean that stock prices are stochastic?
B. What is the distribution of the sum of a large number of random variables of similar size?
C. What is the distribution of the product of a large number of random variables of similar size?
D. What is the assumed distribution of the stock price at expiration of the option?

Part A: Stock prices are uncertain. They may rise or fall; we do not know beforehand.

Question: Lack of knowledge does not make something stochastic. If we knew more about the firm, such as its business strategy and its products, perhaps we could predict if its stock price will rise or fall.

Answer: Thousands of investors study financial statements and other information to determine if a stock price will rise or fall. Everything that can be learned about the firm is considered by investors. If they thought the stock price will rise, they would bid up the stock price now; if they thought it will fall, they would bid it down. What remains the unpredictable things that might cause the stock price to change.

Part B: The central limit theorem says that the sum of a large number of random variables of similar size has a normal distribution.

Part C: Let *a, b, c, …* be the large number of random variables of similar size. Then $ln(a)$, $ln(b)$, $ln(c)$, … are also a large number of random variables of similar size. The central limit theorem says that

$$ln(a) + ln(b) + ln(c) + \ldots \text{ has a } \textit{normal} \text{ distribution, so}$$
$$e^{ln(a) + ln(b) + ln(c) + \cdots} = a \times b \times c \times \ldots \text{ has a } \textit{lognormal} \text{ distribution}$$

Part D: The mean stock price in one year is the expected return times the current price = S × (1 + R%). The future stock price has a lognormal distribution with a mean of S × (1 + R%).

** Exercise 23.4: Assumptions of Black-Scholes

Explain whether each of the following is an assumption of the Black-Scholes formula.

A. Investors in call and put options are not concerned about risk.
B. The expected return on the underlying stock is the same as the risk-free interest rate.
C. The distribution of possible future prices for the underlying stock is lognormal.
D. The option is not yet "in the money."
E. The option is American (i.e., it can be exercised at any time before maturity).

Statement A: False. If the Black-Scholes model applied only to investors who are not concerned about risk, it would not be useful, since almost all investors are concerned about risk.

Statement B: False. From the CAPM, the expected return on the stock is $r_f + \beta \times (E[r_m] - r_f)$. Almost all stocks have ß's greater than zero, so the expected return on the stock is greater than the risk-free rate.

Question: So why does the Black-Scholes formula use the risk-free interest rate?

Answer: The Black-Scholes formula prices options by forming risk-free portfolios consisting of the option minus delta shares of the underlying asset. The risk-free portfolios receive a risk-free rate of return.

Statement C: True. The future value of a stock ranges from zero to infinity. The distribution is not known with certainty, and it is not perfectly lognormal. But we have intuitive reasons to assume a lognormal distribution, which the Black-Scholes formula uses. The Black-Scholes formula appears to give reasonably accurate and unbiased option values (though not perfectly accurate or unbiased), so the lognormal distribution seems a good assumption.

Statement D: False. A call option is in the money if the stock price is greater than the exercise price; a put option is in the money if the stock price is less than the exercise price. The Black-Scholes formula can be used whether the option is in-the-money, out-of-the-money, or at-the-money.

Statement E: False. The Black-Scholes formula prices European options, not American options.