MS Module 4: Hypotheses and Test Procedures – practice problems

(The attached PDF file has better formatting.)

Exercise 4.1: Type I and Type II errors

Let α = the probability of a Type I error, and let β = the probability of a Type II error.

A gambler fears that a coin is biased and is more likely to be Heads than Tails. Let p = the probability that the coin shows up as Heads when it is flipped.

- The null hypothesis is $p \leq 50\%$.
- The alternative hypothesis is $p > 50\%$.

The coin is flipped N = 10 times. The test statistic (X) is the number of Heads in the N coin flips. The null hypothesis is rejected if the number of Heads is more than N–2.

A. What is α, the probability of a Type I error?
B. What is β, the probability of a Type II error, if the true probability of Heads is 60%?

Part A: α = P(0.5, N, N-1) + P(0.5, N, N)

$= n \times ½^{n-1} \times ½ + ½^{n}$
$= 11 / 1{,}024 = 0.010742188$

Part B: We use the 60% probability of Heads for the probability of a Type II error:

β = 1 – P(0.6, N, N-1) – P(0.6, N, N)

$= 1 - n \times 0.6^{n-1} \times 0.4 - 0.6^{n} = 1 - 10 \times 0.6^{9} \times 0.4 - 0.6^{10} = 0.95364$

Exercise 4.2: Type I and Type II errors

Let α = the probability of a Type I error, and let β = the probability of a Type II error.

A gambler uses a biased coin that has a 60% chance of showing Heads when it is flipped. The gambler fears that someone has replaced the biased coin with a fair coin that has a 50% chance of showing Heads when it is flipped. Let p = the probability that the coin is Heads when it is flipped.

- The null hypothesis is p = 60%.
- The alternative hypothesis is p = 50%.

The coin is flipped N = 10 times. The test statistic (X) is the number of Heads in the N coin flips. The null hypothesis is rejected if the number of Heads is less than 3.

A. What is α, the probability of a Type I error?
B. What is β, the probability of a Type II error, if the true probability of Heads is 50%?

Part A: α = P(0.6, N, 0) + P(0.6, N, 1) + P(0.6, N, 2)

$= 0.4^n + n \times 0.4^{n-1} \times 0.6 + \frac{1}{2} \times n \times (n-1) \times 0.4^{n-2} \times 0.6^2$

$= 0.4^{10} + 10 \times 0.4^{(10-1)} \times 0.6 + \frac{1}{2} \times 10 \times (10 - 1) \times 0.4^{(10-2)} \times 0.6^2 = 0.01229$

Part B: We use the 50% probability of Heads for the probability of a Type II error:

β = 1 – (P(0.5, N, 0) + P(0.5, N, 1) + P(0.5, N, 2))

$= 1 - 0.5^n - n \times 0.5^{n-1} \times 0.5 - \frac{1}{2} \times n \times (n-1) \times 0.5^{n-2} \times 0.5^2$

$= 1 - 0.5^{10} - 10 \times 0.5^{(10-1)} \times 0.5 - \frac{1}{2} \times 10 \times (10 - 1) \times 0.5^{(10-2)} \times 0.5^2 = 0.94531$

Exercise 4.3: Probabilities of Type I and Type II errors

A population has a normal distribution with a mean μ_0 of 80 and a standard deviation of 8.

One group from this population has been treated to reduce its mean; we assume it is still normally distributed with the same standard deviation. A sample of size 36 from this treated group has a sample mean of $\bar{x}$ and a true mean of μ'.

- The null hypothesis is H_0: $\mu' = \mu_0$.
- The one-sided alternative hypothesis is H_a: $\mu' < \mu_0$.

We reject the null hypothesis if $\bar{x} \leq 78$.

A. What is the standard deviation of the sample mean?
B. What is the *z* statistic value to test the null hypothesis?
C. What is the probability of a Type I error for this one-sided (lower-tailed) test?
D. If the true mean of the sample is 76, what is the probability of a Type II error for this test?

Part A: The standard deviation of the sample mean is $8 / 36^{0.5} = 1.33333$

Part B: The *z* value is $(78 – 80) / 1.333333 = -1.500000$

Part C: The probability of a Type I error for the one-sided (lower-tailed) test is $\Phi(–1.5) = 0.0668$.

Part E: The probability of a Type II error for the one-sided (lower-tailed) test is

$1 – \Phi((\bar{x} – \mu') / \sigma) = 1 – \Phi ((78 – 76) / 1.33333) = 1 – \Phi(1.5) = 1 – 0.93319 = 0.06681$.

Question: This exercise doesn't give a significance level such as 5% or 1%.

Answer: This exercise gives a rejection region of $\bar{x} \leq 78$. The implied significance level α is 6.681%. We can derive the significance level from the rejection region or the rejection region from the significance level.

Given the rejection region, we derive the probability of a Type II error (β) for each true value of μ' (the true value of the tested sample). The β value can also be expressed in terms of the significance level α, which is the formula used in most examples.

Exercise 4.4: Type 1 and Type 2 errors

Let μ be the unknown mean of a normal distribution with σ = 1,500.

- The null hypothesis is H_0: μ = 30,000
- The alternative hypothesis is H_a: μ > 30,000

A statistician tests the null hypothesis with α (the probability of a Type I error) = 1% and a sample of size 16.

A. What is the probability of making a type II error if μ = 31,000?
B. If β (the probability of a type II error when μ = 31,000) is to be no more than 10%, how many observations N are needed?

Part A: A test with $\alpha = 0.01$ requires $z_a = z_{.01} = 2.33$. The probability of making a type II error if $\mu = \mu'$ is

$\Phi(z_\alpha + (\mu_0 - \mu') / (\sigma/\sqrt{n}))$

so $\beta(31{,}000) = \Phi(2.33 + (30{,}000 - 31{,}000) / (1500 / \sqrt{16})) = \Phi(-0.34) = 0.3669$

so $\beta(31{,}000) = \Phi(2.32635 + (30{,}000 - 31{,}000) / (1500 / \sqrt{16})) = \Phi(-0.3403) = 0.3668$

$2.32635 + (30{,}000 - 31{,}000) / (1500 / 16^{0.5}) = -0.3403$

(Excel: =NORM.DIST(-0.34,0,1,1))
(Excel: =NORM.DIST(-0.3403,0,1,1))

Part B: the needed sample size is

$n = (\sigma \times (z_\alpha + z_\beta) / (\mu_0 - \mu'))^2$

If $\beta(31{,}000) = 0.1$, so $z_{0.1} = 1.28$, the needed sample size is

$n = (1{,}500 \times (2.33 + 1.28) / (30{,}000 - 31{,}000))^2 = (-0.542)^2 = 29.32.$

$n = (1{,}500 \times (2.32635 + 1.28155) / (30{,}000 - 31{,}000))^2 = 29.29$

Exercise 4.5: Type 1 and Type 2 errors, one-tailed test

Let μ be the unknown mean of a normal distribution with $\sigma = 18$.

- The null hypothesis is H_0: $\mu = 100$
- The alternative hypothesis is H_a: $\mu > 100$

A statistician tests the null hypothesis with α (the probability of a Type I error) = 1% and a sample of size 36.

A. What is the probability of making a type II error if $\mu = 110$, or $\beta(110)$?
B. If $\beta(110)$ is to be no more than 10%, how many observations N are needed?

Part A: A test with $\alpha = 0.01$ requires $z_\alpha = z_{.01} = 2.32635$. The probability of making a type II error if $\mu = \mu'$ is

$\Phi(z_\alpha + (\mu_0 - \mu') / (\sigma/\sqrt{n}))$

We compute: $2.32635 + (100 - 110) / (18 / 36^{0.5}) = -1.006983$

so $\beta(110) = \Phi(2.32635 + (100 - 110) / (18 / \sqrt{36})) = \Phi(-1.006983) = 0.15697$

For the final exam, you look up the values of the cumulative standard normal distribution in tables provided. For the practice problems, values of the cumulative standard normal distribution are given by the Excel built-in function NORM.S.DIST, and values for other normal distributions are given by the function NORM.DIST.

Part B: the needed sample size is $n = (\sigma \times (z_\alpha + z_\beta) / (\mu_0 - \mu'))^2$

If $\beta(110) = 0.1$, $z_{0.1} = 1.28155$, and the needed sample size is

$$n = (18 \times (2.32635 + 1.28155) / (100 - 110))^2 = 42.17$$

The number of observations is an integer. For $\beta(110)$ to be *no more than* 10%, we round up to 43.

Exercise 4.6: Type 1 and Type 2 errors, two-tailed test

Let μ be the unknown mean of a normal distribution with $\sigma = 18$.

- The null hypothesis is H_0: $\mu = 100$
- The alternative hypothesis is H_a: $\mu \neq 100$

A statistician tests the null hypothesis with α (the probability of a Type I error) = 1% and a sample of size 36.

A. What is the probability of making a type II error if $\mu = 110$?
B. If $\beta(110)$, the probability of a type II error when $\mu = 110$, is to be no more than 10%, how many observations N are needed?

Part A: A test with $\alpha = 0.01$ requires $z_{\alpha/2} = z_{.005} = 2.57583$. The probability of making a type II error if $\mu = \mu'$ is

$\Phi(z_{\alpha/2} + (\mu_0 - \mu') / (\sigma/\sqrt{n})) - \Phi(-z_{\alpha/2} + (\mu_0 - \mu') / (\sigma/\sqrt{n}))$

The formula for the two-sided test replaces z_α by $z_{\alpha/2}$; for z = 0.01, z_α = 2,326348, and $z_{\alpha/2}$ = 2.575829.

We compute:

- $2.57583 + (100 - 110) / (18 / 36^{0.5}) = -0.75750$
  - $\Phi(-0.75750) = 0.224375$
- $-2.57583 + (100 - 110) / (18 / 36^{0.5}) = -5.90916$
  - $\Phi(-5.90916) = 1.7193\text{E-}09$ (≈ 0).

so $\beta(110) = \Phi(-0.75750) - \Phi(-5.90916) = 0.224375$

For the final exam, you look up the values of the cumulative standard normal distribution in tables provided. For the practice problems, values of the cumulative standard normal distribution are given by the Excel built-in function NORM.S.DIST, and values for other normal distributions are given by the function NORM.DIST.

Part B: The needed sample size for a two-tailed test is $n = (\sigma \times (z_{\alpha/2} + z_\beta) / (\mu_0 - \mu'))^2$

If $\beta(110) = 0.1$, $z_{0.1} = 1.28155$, and the needed sample size is

$$n = (18 \times (2.57583 + 1.28155) / (100 - 110))^2 = 48.21$$

Exercise 4.7: *T* test, one sample

A sample of N=15 observations from a normal distribution has

- Σx_i = 120 (sum of the N observations)
- Σx_i^2 = 3,600 (sum of the squares of the N observations)

- The null hypothesis is H_0: the population mean $\mu_0 = 4$
- This exercise shows the procedure for one-tailed and two-tailed tests, with either
  - The two-tailed alternative hypothesis is H_a: the population mean $\mu_0 \neq 4$
  - The one-tailed (upper tailed) alternative hypothesis is H_a: the population mean $\mu_0 > 4$

A. What is the sample mean of the N observations?
B. What is the sample standard deviation of the N observations?
C. What is the *t* value used to test the null hypothesis?
D. What is the *p* value for the one-tailed alternative hypothesis?
E. What is the *p* value for the two-tailed alternative hypothesis?

Part A: The sample mean is Σx_i / N = 120 / 15 = 8.

Part B: The sample standard deviation = $\{ [\Sigma x_i^2 - (\Sigma x_i)/N] / (N - 1) \}^{0.5}$ =

$$((3{,}600 - 120^2 / 15) / (15 - 1))^{0.5} = 13.732131$$

Part C: The *t* value used to test the null hypothesis is $(\bar{x} - \mu_0) / (\sigma / N^{0.5})$ =

$$(8 - 4) / (13.732131 / 15^{0.5}) = 1.128152$$

Part D: The *p* value for the one-tailed alternative hypothesis = 0.1391 (table look-up or spreadsheet function)

Part E: The *p* value for the two-tailed alternative hypothesis = 0.2782 (table look-up or spreadsheet function)