MS Module 5 *t* values and confidence intervals practice problems

(The attached PDF file has better formatting.)

Exercise 5.1: *T* test, one sample

A sample of N=15 observations from a normal distribution has

! Σx_i = 120 (sum of the N observations)
! Σx_i^2 = 3,600 (sum of the squares of the N observations)

! The null hypothesis is H_0: the population mean $\mu_0 = 4$
! This exercise shows the procedure for one-tailed and two-tailed tests, with either
  " The two-tailed alternative hypothesis is H_a: the population mean $\mu_0 \neq 4$
  " The one-tailed (upper tailed) alternative hypothesis is H_a: the population mean $\mu_0 > 4$

A. What is the sample mean of the N observations?
B. What is the sample standard deviation of the N observations?
C. What is the *t* value used to test the null hypothesis?
D. What is the *p* value for the one-tailed alternative hypothesis?
E. What is the *p* value for the two-tailed alternative hypothesis?

Part A: The sample mean is Σx_i / N = 120 / 15 = 8.

Part B: The sample standard deviation = $\{ [\Sigma x_i^2 - (\Sigma x_i)/N] / (N - 1) \}^{0.5}$ =

$$((3{,}600 - 120^2 / 15) / (15 - 1))^{0.5} = 13.732131$$

Part C: The *t* value used to test the null hypothesis is $(\bar{x} - \mu_0) / (\sigma / N^{0.5})$ =

$$(8 - 4) / (13.732131 / 15^{0.5}) = 1.128152$$

Part D: The *p* value for the one-tailed alternative hypothesis = 0.1391 (table look-up or spreadsheet function)

Part E: The *p* value for the two-tailed alternative hypothesis = 0.2782 (table look-up or spreadsheet function)

Exercise 5.2: *t* values and confidence intervals

A statistician estimates the population mean for a normal distribution from a sample of 8 points. The 99% confidence interval for the population mean is (0, 2.500)

A. What is the critical *t* value for a 99% confidence interval from a sample of 8 points?
B. What is the standard deviation of the sample?
C. What is the critical *t* value for a 95% confidence interval from a sample of 8 points?
D. What is the 95% confidence interval for the population mean?

Part A: A sample with 8 points has 7 degrees of freedom. The critical *t* value for a 99% confidence interval with 7 degrees of freedom is 3.499 (table lookup).

Part B: The confidence interval is the estimate ± σ/√N × *t* value, so

! the point estimate is (2.500 + 0) / 2 = 1.250
! the width of the confidence interval is 2 × σ/√N × *t* value = 2.500 – 0 = 2.500 ⇛
  " σ = $8^{0.5}$ × 2.500 / (2 × 3.499) = 1.010

Part C: The critical *t* value for a 95% confidence interval with 7 degrees of freedom is 2.365 (table lookup).

Part D: The confidence interval is 1.250 ± (1.010 / $8^{0.5}$) × 2.365:

! lower bound: 1.250 – (1.010 / $8^{0.5}$) × 2.365 = 0.405
! upper bound: 1.250 + (1.010 / $8^{0.5}$) × 2.365 = 2.095

Exercise 5.3: *t* values and two confidence intervals

A statistician estimates the population mean for a normal distribution from a sample of 8 points.

! The upper bound of the 95% confidence interval for the population mean is 5.
! The lower bound of the 90% confidence interval for the population mean is 1.

We use the following notation:

! μ = the estimated population mean
! σ = the standard deviation of the sample
! N = the number of observations

A. What is the critical *t* value for a two-sided 95% confidence interval for a sample of 8 points?
B. What is the critical *t* value for a two-sided 90% confidence interval for a sample of 8 points?
C. What is $\sigma/\sqrt{N}$?
D. What is σ?
E. What is μ?

Part A: A sample of 8 points has 8 – 1 = 7 degrees of freedom.

The critical *t* value for a two-sided 95% confidence interval for a sample of 8 points is 2.3646 (table look-up).

Part B: A sample of 8 points has 8 – 1 = 7 degrees of freedom.

The critical *t* value for a two-sided 90% confidence interval for a sample of 8 points is 1.8946 (table look-up).

Part C: Combine the upper half of the 95% confidence interval with the lower half of the 90% confidence interval to estimate $\sigma/\sqrt{N}$:

! $2.3646 \times \sigma/\sqrt{N} = 5 - \mu$
! $1.8946 \times \sigma/\sqrt{N} = \mu - 1$

adding: $(2.3646 + 1.8946) \times \sigma/\sqrt{N} = (5 - \mu) + (\mu - 1) = (5 - 1) \Rrightarrow$

$\sigma/\sqrt{N} = (5 - 1) / (2.3646 + 1.8946) = 0.93914$

Part D: $\sigma = 0.93914 \times 8^{0.5} = 2.656$

Part E: Solve for μ by either of the confidence intervals:

! $2.3646 \times \sigma/\sqrt{N} = 5 - \mu \Rrightarrow \mu = 5 - 2.3646 \times 0.93914 = 2.779$
! $1.8946 \times \sigma/\sqrt{N} = \mu - 1 \Rrightarrow \mu = 1 + 1.8946 \times 0.93914 = 2.779$