Corporate Finance, Module 21: “Option Valuation”

Practice Problems: (representative of the final exam problems)

(The attached PDF file has better formatting.)

EXERCISE 21.1: BINOMIAL TREE PRICING METHOD

A firm’s current share price is $25.00; one year from now, its share price will either fall to $20.00 or rise to $31.25. The risk-free rate is 6%, and a one-year European call option on the stock has an exercise price of $30.00.

A. What is the upward movement if the stock price rises to $31.25?
B. What is the value of the call option if the stock price rises to $31.25?
C. What is the downward movement if the stock price falls to $20.00?
D. What is the value of the call option if the stock price falls to $20.00?
E. What is the risk-neutral probability of a rise in the stock price?
F. What is the expected value of the call option at expiration in a risk-neutral world?
G. What is the present value of the call option?

Solution 21.1:

Part A: The stock price is now $25, so $31.25 is an increase of 25% (a factor of 1.25).

Part B: The call option is worth $31.25 – $30.00 = $1.25 if the stock price rises.

Part C: A decline to $20 is a decrease of 20% (a factor of 0.80).

Part D: The call option expires worthless if the stock price declines.

Part E: Since the risk-free rate is 6%, the risk-neutral probability of a rise in the stock price is (1.06 – 0.80) / (1.25 – 0.80) = 57.78%.

Part F: The expected value of the call option in a risk-neutral world one year from now is 57.78% × ($31.25 – $30.00) + (1 – 57.78%) × $0 = $0.722.

Part G: The present value of the call option is $0.722 / 1.06 = $0.681.

Exercise 21.2: Options and Interest Rates

A stock with a share price now of $20 will have a price of either $14 or $30 at the end of one year. A one-year call option with exercise price of $26 sells for $2.

A. What is the value of the call option at expiration if the stock price rises to $30?
B. What is the value of the call option at expiration if the stock price falls to $14?
C. What is the value of the call option delta?
D. What is the risk-free portfolio of shares minus a call option?
E. What is the current value of this risk-free portfolio?
F. What is the value of this risk-free portfolio at expiration of the stock price rises to $30?
G. What is the value of this risk-free portfolio at expiration of the stock price falls to $14?
H. What return does the risk-free portfolio offer?
I. What is the risk-free rate?
J. What is the upward movement in the stock price if the price rises to $30?
K. What is the downward movement in the stock price if the price falls to $14?
L. What is the risk-neutral probability of a rise in the stock price?
M. What is the expected value of the call option in the risk-neutral world at expiration?
N. What is the risk-free rate?

Solution 21.2:

We can solve this with either the risk neutral valuation method or the option delta method. We show first the option delta method and then the risk neutral valuation method.

Part A (Option Delta method): The call option is worth $30 – $26 = 4 if the stock price rises to $30.

Part B: The call option is worth $0 if the stock price declines to $14.

Part C: The call option delta is ($4 – $0) / ($30 – $14) = $4/$16 = 25%.

Part D: The risk free portfolio is *delta shares of stock* minus one call option (or "S/4 – c").

Part E: At the present time, the stock is worth $20 and the call option is worth $2, so the risk-free portfolio "S/4 – c" is worth $20/4 – $2 = $3.

Part F: One year from now, the risk-free portfolio "S/4 – c" will be worth $30 / 4 – $4 = $3.50 if the stock price rises to $30.

Part G: One year from now, the risk-free portfolio "S/4 – c" will be worth $14 / 4 – $0 = $3.50 if the stock price declines to $14.

Part H: The portfolio is risk-free, so it must earn the risk-free return.

Part I: We write

$$\$3 \times (1 + r_f) = \$3.50$$
$$r_f = \$3.50 / \$3.00 - 1 = 16.67\%.$$

Part J (Risk Neutral Valuation method): The upward movement is $30 / $20 = 1.500.

Part K: The downward movement is $14 / $20 = 0.700.

Part L: The risk-neutral probability of a rise in the stock price is $p = (1 + r_f - 0.700) / (1.500 - 0.700)$.

Part M: The value of the call option in a risk-neutral world one year from now is p × ($30 – $26) + (1 – p) × $0 = p × $4.

Part N: The present value of the call option is p × $4 / (1 + r_f). We write

$$p \times \$4 / (1 + r_f) = \$2$$
$$(1 + r_f - 0.700) / (1.500 - 0.700) \times \$4 / (1 + r_f) = \$2$$
$$(1 + r_f - 0.700) = 1.600 \times (1 + r_f)$$
$$r_f = 0.700 / 0.600 - 1 = 16.67\%.$$

Exercise 21.3: European Call Option

A stock now trades at $62.50. The price will rise by 50% or fall by one third over the next year. The exercise price of a one year European call option is $83 a ($83.33). The risk-free interest rate is 10% per annum.

A. What is the stock price if it rises 50%?
B. What is the value of the European call option if the stock price rises 50%?
C. What is the stock price if it falls by one third?
D. What is the value of the European call option if the stock price falls by one third?
E. What is the call option delta?
F. What is the risk-neutral probability of a rise in the stock price?
G. What is the expected value of the call option at expiration in a risk-neutral world?
H. What is the present value of the call option?

Solution 21.3:

Part A: The stock price is $62.50 × 1.50 = $93.75 if the stock price rises 50%.

Part B: The value of the European call option is $93.75 – $83.33 = $10.42 if the stock price rises 50%.

Part C: The stock price is $62.50 × b = $41.67 if the stock price declines by one third.

Part D: The value of the European call option is zero if the stock price falls by a third.

Part E: The option delta is ($10.42 – $0) / ($93.75 – $41.67) = 20%.

Part F: The risk-neutral probability of a rise in the stock price is (1.100 – 0.667) / (1.500 – 0.667) = 52%.

Part G: If the stock price rises to $93.75, the call option will be worth $93.75 – $83.33 = $10.42; if the stock price falls to $41.67, the call option will be worth zero. The expected value of the call option one year from now in a risk-neutral world is 52% × $10.42 = $5.42.

Part H: The present value of the call option is $5.42 / 1.100 = $4.93.

Exercise 21.4: Options and Interest Rates

A one-year American put option on a non-dividend paying stock trading for $100 has an exercise price of $105. The stock price will increase by 10% or decrease by 15% by the end of the year. The investor is indifferent between exercising now or waiting one year.

We can solve this with the option delta method or the risk neutral valuation method. We show first the option delta method and then the risk neutral valuation method.

A. What is the payoff if the American put option is exercised now?
B. If the stock price rises by 10%, what is the value of the put option?
C. If the stock price declines by 15%, what is the value of the put option?
D. What is the delta of the put option?
E. What combination of a put option plus some shares is a risk-free portfolio?
F. What is the present value of this risk-free portfolio?
G. What is the value of this risk-free portfolio at expiration if the stock price rises?
H. What is the value of this risk-free portfolio at expiration if the stock price falls?
I. What is the return earned on the risk-free portfolio?
J. What is the risk-free interest rate (r_f)?
K. What is the risk-neutral probability of a rise in the stock price in terms of r_f?
L. What is the value of the put option in a risk-neutral world at expiration in terms of r_f?
M. What is the present value of the put option in terms of r_f?
N. What is the risk-free interest rate (r_f)?

Solution 21.4:

Part A. Option Delta Method: If we exercise the put option now, the payoff is $105 – $100 = $5.

Part B: If the stock price rises to $110, the put option is worth zero.

Part C: If the stock price falls to $85, the put option is worth $105 – $85 = $20.

Part D: The delta for the put option is ($0 – $20) / ($110 – $85) = –80%.

Part E: The combination of the put option *minus* Δ *stock shares* is a risk-free portfolio; this is 1 put – (–0.8) shares = 1 put + 0.8 shares.

Part F: At the present time, the value of this portfolio is 1 × $5 + 0.8 × $100 = $85.

Part G: In one year, the portfolio is worth 0.8 × $110 = $88 if the stock price rises.

Part H: In one year, the portfolio is worth 0.8 × $85 + $20 = $88 if the stock price falls; the value is the same $88, since the portfolio is risk-free.

Part I: The return must be the risk-free rate if the portfolio is risk-free.

Part J: We have $85 × (1 + r_f) = $88 ⇛ r_f = $88 / $85 – 1 = 3.53%.

Part K – Risk Neutral Valuation method: The upward movement is $110 / $100 = 1.100 and the downward movement is $85 / $100 = 0.850. The risk-neutral probability that the stock price rises is $p = (1 + r_f – 0.850) / (1.100 – 0.850)$.

Park L: The value of the put option in a risk-neutral world one year from now is p × ($0) + (1 – p) × ($105 – $85) = (1 – p) × $20.

Part M: The present value of the put option is $(1 – p) \times \$20 / (1 + r_f)$.

Part N: We write

$$(1 - p) \times \$20 / (1 + r_f) = \$5$$
$$(1 - p) \times 4 = 1 + r_f$$
$$4 \times (0.1 - r_f) / 0.25 = 1 + r_f$$
$$16 \times (0.1 - r_f) = 1 + r_f$$
$$1.6 - 16r_f = 1 + r_f$$
$$0.6 = 17r_f$$
$$r_f = 0.6 / 17 = 3.53\%$$

Exercise 21.5: Value of Waiting

You own a one-year American put option on a non-dividend paying stock with an exercise price of $42. The stock currently sells at $40, and is expected to sell at either $39 or $48 one year from now. What annual risk-free interest rate would equate the value of exercising the put immediately with the present value of waiting until the end of the term?

Solution 21.5: We can solve this with either the risk neutral valuation method or the option delta method. We show both methods.

Option Delta Method: If we exercise the put option now, the payoff is $42 – $40 = $2. If the stock price rises to $48, the put option is worth zero; if the stock price falls to $39, the put option is worth $42 – $39 = $3. The delta for the put option is ($0 – $3) / ($48 – $39) = – a. The combination of the put option minus Δ shares of the stock is a risk-free portfolio; this is 1 put + a stock. At the present time, the value of this portfolio is 1 × $2 + a × $40 = $15 a. In one year, the portfolio is worth a × $48 = $16 if the stock price rises and a × $39 + $3 = $16 if the stock price falls; the value is the same $16, since the portfolio is risk-free. The return must be the risk-free rate if the portfolio is risk-free, so we have $15 a × (1 + r_f) = $16 ⇛ r_f = $16 / $15 a – 1 = 4.35%.

Risk Neutral Valuation method: The upward movement is $48 / $40 = 1.200 and the downward movement is $39 / $40 = 0.975. The risk-neutral probability of a rise in the stock price is *p* = (1 + r_f – 0.975) / (1.200 – 0.975). The value of the put option in a risk-neutral world one year from now is p × ($0) + (1 – p) × ($42 – $39) = (1 – p) × $3. The present value of the put option is (1 – p) × $3 / (1 + r_f). We write

$$(1 - p) \times \$3 / (1 + r_f) = \$2$$
$$(1 - p) \times 1.500 = 1 + r_r$$
$$1.500 \times (0.2 - r_f) / 0.225 = 1 + r_f$$
$$6.667 \times (0.2 - r_f) = 1 + r_f$$
$$7.667 \times r_f = 0.333$$
$$r_f = 0.333 / 7.667 = 4.34\%$$

Exercise 21.6: Put-Call Option

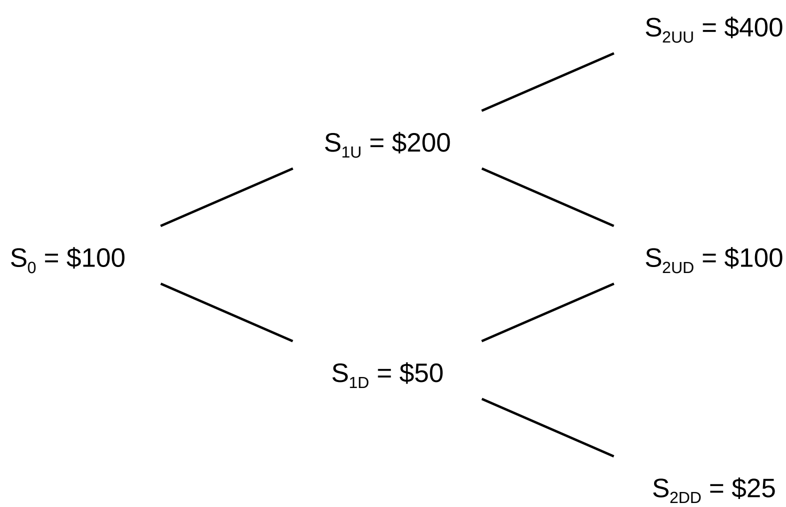
The price of a non-dividend paying stock is $100 and could halve or double in each six-month period. The risk-free interest rate is 10.25% a year. An option allows the investor to either sell the stock in month 6 for $75 or buy it in month 12 for $75.

A. Draw a two period binomial tree showing the stock price at each node.
B. If the stock price declines in the first period to $50, what should the investor do?
C. If the stock moves up in the first period to $200, should the investor exercise the option or hold on to it exercise it in the second period?
D. If the stock price moves up in the first period and again in the second period to $400, what should the investor do?
E. If the stock price moves up in the first period and then down in the second period to $100, what should the investor do?
F. What is the risk-free interest rate for a half-year period?
G. What is the risk-neutral probability of a rise in the stock price?
H. What is the present value of the option?

Solution 21.6:

Part A: We draw a two period binomial tree showing the stock price at each node:

Figure 1: Stock Price Binomial Lattice



Part B: If the stock price declines in the first period to $50, we sell the stock for $75; the option is worth $25.

Part C: If the stock moves up in the first period to $200, we hold the option, and we exercise it in the second period.

Part D: If the stock price moves up again to $400, we buy the stock for $75; the option is worth $325.

Part E: If the stock price moves down to $100, we buy the stock for $75; the option is worth $25.

Part F: The risk-free rate is 10.25% for the year or 5.00% each half year.

Part G: The upward movement is 2.00 (or +100%); the downward movement is 0.50 (or –50%). The risk-neutral probability of an upward movement in the stock price is (1.05 – 0.50) / (2.00 – 0.50) = 0.55 / 1.50 = 36.67%.

Part H: We derive the present value of the option as follows:

- The risk-neutral probability of getting to S_{1D} (moving down in the first period) is 1 – 36.67% = 63.33%.
- The risk-neutral probability of getting to S_{2UU} (moving up in both the first and second periods) is $36.67\%^2$ = 13.44%.
- The risk-neutral probability of getting to S_{2UD} (moving up in the first period and down in the second period) is 36.67% × (1 – 36.67%) = 23.22%.

We multiply the value of the option at each node by the risk-neutral probability of getting to that node, and we discount back to the present time at the risk-free rate. From the first period node S_{1D} we discount for one period; the discount factor is 1/1.05. From the second period nodes S_{2UU} and S_{2UD}, we discount for two periods; the discount factor is $1/1.05^2$ = 1.1025.

The value of the option is

	63.33% × \$25 / 1.05	=	\$15.08
+	13.44% × \$325 / 1.1025	=	\$39.62
+	23.22% × \$25 / 1.1025	=	\$5.27
=			\$59.98

Exercise 21.7: Probability of Increase

Assume a stock price can either rise by 28.4% or decrease by 22.1% over a one-year period, and that the annual effective risk-free interest rate is 16%. The market risk premium is 8%, and the stock's CAPM beta is 1.25.

A. What is the expected return on the stock? (Use the Capital Asset Pricing Model)
B. Write an expression for the expected return on the stock in terms of *P*, the *actual* probability of a rise in the stock price.
C. What is the *actual* probability of a rise in the stock price?
D. What is the expected return on the stock in a risk-neutral world?
E. Write an expression for the expected return on the stock in terms of P', the *risk-neutral probability* of a rise in the stock price.
F. What is the *risk-neutral probability* of a rise in the stock price?

Solution 21.7:

For the true probability of increase, we want the expected return on the stock; for the risk-neutral probability of increase, we want the risk-free rate.

Part A: The return on the stock (using the CAPM) is 16% + 1.25 × 8% = 26%.

Part B: The equation for the expected return is $P \times 1.284 + (1 - P) \times 0.779 = 1.260$

Part C: We solve: $P = (1.26 - 0.779) / (1.284 - 0.779) = 95.25\%$

Part D: The expected return in a risk-neutral world is the risk-free rate of 16%.

Part B: The equation for the expected return is $P' \times 1.284 + (1 - P) \times 0.779 = 1.160$

Part C: We solve: $P' = (1.16 - 0.779) / (1.284 - 0.779) = 75.45\%$

Exercise 21.8: One Year Call Option

A stock is currently trading at \$70. Over the next year, the stock can either appreciate by 25% or depreciate by 20%. A one-year call option on the stock has an exercise price of \$70. The annual risk-free rate is 7%.

A. What is the risk-neutral probability of a rise in the stock price?
B. What is the value of the option at expiration if the stock price rises 25%?
C. What is the value of the option at expiration if the stock price falls 20%?
D. What is the expected value of the option at expiration in the risk-neutral world?
E. What is the present value of the call option?
F. What is the call option delta?
G. What is the risk-free portfolio of call options and stock shares?
H. What is the value of this risk-free portfolio? (We treat this value as a risk-free bond.)
I. What combination of shares and a risk-free loan replicates the call option? (Rewrite the value of the risk-free portfolio as 1 call option = Y shares ± Z dollars of bonds.)

Solution 21.8:

Part A: The risk-neutral probability of a rise in the stock price is

$$r = (1.07 - 0.80) / (1.25 - 0.80) = 60.00\%$$

Part B: If the stock price rises to \$70 × 1.25 = \$87.50, the call option is worth \$87.50 – \$70 = \$17.50.

Part C: If the stock price falls to \$70 × 0.80 = \$56, the call option is worth zero.

Part D: The expected value of the call option at expiration in the risk-neutral world is 60% × (\$87.50 – \$70) = \$10.50.

Part E: The present value of the call option is 60% × (\$87.50 – \$70) / 1.07 = \$9.81.

Part F: The call option delta is the change in the price of the call option divided by the change in the price of the underlying security. More precisely, the call option delta is the partial derivative of the call option price with respect to the price of the underlying security, but we use the ratio as the call option delta in the binomial tree pricing problems. The call option delta in this exercise is

$$(\$17.50 - \$0) / (\$87.50 - \$56) = 17.50 / 31.50 = 55.56\%$$

Part G: A portfolio of 0.5556 shares – 1 call option is risk-free.

Part H: This portfolio costs 0.5556 × \$70 – \$10.09 = \$28.80.

Part I: A portfolio of 0.5556 shares less \$28.80 of a risk-free loan replicates the call option:

$$1 \text{ call option} = 0.5556 \text{ shares} - \$28.80$$

Question 21.9: Option Delta

If x = the call option delta of a stock, which of the following is a risk-free portfolio?

A. Sell x shares, sell a call, and borrow the balance.
B. Sell x shares, buy a call, and borrow the balance.
C. Buy x shares, sell a call, and borrow the balance.
D. Buy x shares, buy a call, and borrow the balance.
E. None of these positions has an expected value of $0.

Solution: C

As the stock price increases, the call option price increases. The option delta is the ratio of the call option price change to the stock price change:

$$\Delta = \text{change in call option price / change in stock price}$$

To create a risk-free portfolio, we must either buy shares and sell call options or sell shares and buy call options; these are choices B and C.

Which costs more: one call option or delta shares of the stock? We use a numerical example to show the reasoning. Suppose a stock costs $100 and will either increase to $110 or decrease to $90 over the next period. The call option exercise price is $105. The call option delta is the change in the call option price divided by the change in the stock price, or [($110 – $105) – $0] / [$110 – $90] = $5 / $20 = 25%.

To price the call option, we must know the risk-free interest rate. For simplicity, we use a risk-free rate of a 5% effective annual yield. The risk-neutral probability of a rise in the stock price is (1.05 – 0.90) / (1.10 – 0.90) = 0.15 / 0.20 = 75%. The price of the call option is 75% × $5 / 1.05 = $3.57.

Delta shares of the stock are $25; one call option costs $3.57. If we buy delta shares of the stock and sell the call option, we *borrow* the difference.