Corporate Finance, Module 21: “Option Valuation”

Practice Problems

(The attached PDF file has better formatting.)

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{This posting has more information than is needed for the corporate finance on-line course. The past Course 2 exam problems on options pricing have been difficult, covering a wide range of topics. Most of the examples in this posting are for candidates taking the CAS transition exam, who may be confronted with problems like those shown here.}

Binomial Tree Valuation

Jacob: I can’t get the hang of these option pricing formulas.

Rachel: The first several times that you work with binomial tree valuation, you should draw the tree and work out the risk-free portfolio for both calls and puts. Focus on two items:

- The characteristics of the option delta – why it is positive for calls and negative for puts, and why it changes over time.
- The risk-neutral probability measure, which doesn’t change over time as long as the *up* and *down* parameters don’t change.

Question 21.1: Option Delta

Which of the following statements is true regarding option deltas?

A. The delta of a stock option is the ratio of the change in the price of a stock option to the change in the price of the stock itself (the underlying security).
B. The delta is the number of options that we should hold for each stock shorted to create a riskless hedge.
C. The delta of a call option is negative, whereas the delta of a put option is positive.
D. The delta of the call option plus the delta of the corresponding put option (with the same expected and exercise price) is one.
E. A higher option delta indicates that the option is riskier.

Answer 21.1: A

The option delta is the *partial derivative* of the option price with respect to the stock price.

For binomial tree pricing, the formula for the option delta is the discrete analogue of the partial derivative. Let S_0 be the current stock price and f be the derivative security:

u is the upward movement in the stock price
d is the downward movement in the stock price
f_u is the value of the option if the stock price moves up (sometimes denoted f^+)
f_d is the value of the option if the stock price moves down (sometimes denoted f^-)

The superscripts + and – are sometimes used for the stock, the option, the call, and the put:

S^+, f^+, c^+, p^+ and S^-, f^-, c^-, p^-

The option delta is

$$\text{option } \Delta = (f_u - f_d)/(S_0 \times u - S_0 \times d)$$

Jacob: Are u and d factors, such as 1.150 and 0.850, or percentage changes, such as +15% and –15%?

Rachel: Either definition is fine. We use the difference between u and d. If we use factors, we get 1.150 – 0.850 = 0.300. If we use percentage changes, we get 15% – (–15%) = 30% = 0.300. When we use the risk-neutral probabilities, we must be consistent between stock price movements and interest rates. If u and d are factors, the interest rate is a factor, like 1.08; if u and d are percentage changes, the interest rate is a percentage, such as 8%.

Statement B is incorrect; it should say:

> *The delta is the shares of stock we should hold for each option shorted to create a riskless hedge.*

If we hold delta (Δ) shares of stock and we short one option, the portfolio is

$$\Delta S - f.$$

You will sometimes see this risk-free portfolio as $f - \Delta S$, since if a portfolio if risk-free, its negative is risk-free.

At the end of the period, there are two possibilities: the *up* state and the *down* state. If the up state occurs, S_0 becomes $S_0 \times u$ and *f* becomes f_u, so the value of the portfolio is

$$\Delta \times S_0 \times u - f_u$$

We replace Δ by $(f_u - f_d) / (S_0 \times u - S_0 \times d)$ to get

$$\Delta \times S_0 \times u - f_u = (f_u - f_d) / (S_0 \times u - S_0 \times d) \times S_0 \times u - f_u$$

If the down state occurs, S_0 becomes $S_0 \times d$ and *f* becomes f_d, so the value of the portfolio is

$$\Delta \times S_0 \times u - f_d = (f_u - f_d) / (S_0 \times u - S_0 \times d) \times S_0 \times d - f_d$$

A riskless portfolio has the same value in all states of the world, whether the stock price moves up or down, so if the portfolio is riskless, these two values must be the same:

$$(f_u - f_d) / (S_0 \times u - S_0 \times d) \times S_0 \times u - f_u = (f_u - f_d) / (S_0 \times u - S_0 \times d) \times S_0 \times d - f_d$$

We show this algebraically. We factor out the S_0's to get

$$(f_u - f_d) / (u - d) \times u - f_u = (f_u - f_d) / (u - d) \times d - f_d$$

We add f_u to each side to get

$$(f_u - f_d) / (u - d) \times u = (f_u - f_d) / (u - d) \times d + (f_u - f_d)$$

We factor out the $(f_u - f_d)$ to get

$$u / (u - d) = d / (u - d) + 1$$
$$(u - d) / (u - d) = 1$$

Delta Limits

Statement C should be reversed:

- For a *call* option, when the price of the underlying security (the stock) increases, the payoff from the call option increases. Since the two changes are positively correlated, the ratio of the two changes is *positive*.
- For a *put* option, when the price of the underlying security (the stock) increases, the payoff from the put option decreases. Since the two changes are negatively correlated, the ratio of the two changes is *negative*.

Statement D should say that the call option delta equals the put option delta plus one. The put call parity relation says that c + PV(X) = p + S. We take the partial derivative of both sides with respect to the stock price S.

- $\partial c/\partial S$ is the call option delta.
- $\partial p/\partial S$ is the put option delta.
- $\partial PV(X)/\partial S$ is zero.
- $\partial S/\partial S$ is one.

Statement E should say that an option is riskiest when its delta is near ½ or –½.

- If a call option is deep in the money, meaning that the stock price is much higher than the exercise price, the option will almost surely be exercised, and it is not risky.
- If a call option is way out of the money, meaning that the stock price is much lower than the exercise price, the option will almost surely not be exercised, and it is not risky.
- If a put option is deep in the money, meaning that the stock price is much lower than the exercise price, the option will almost surely be exercised, and it is not risky.
- If a call option is way out of the money, meaning that the stock price is much higher than the exercise price, the option will almost surely not be exercised, and it is not risky.

If the option has a delta of about ½, it may or may not be exercised, and it is risky.

Jacob: Suppose the exercise price is 50, and there is one day left to expiration.

- If the stock price is 100 and the call option delta is about one, a $1 change in the stock price causes about a $1 change in the call option price.
- If the stock price is 50 and the call option delta is about ½, a $1 change in the stock price causes about a $0.50 change in the call option price.

The first scenario has a larger change in the call option price.

Rachel: You must also consider the present price of the option.

- If the stock price is 100 and the call option delta is about one, the call option price is about $50, and a $1 change in the stock price causes 2% change in the call option price.

- If the stock price is 50 and the call option delta is about ½, the call option price is about \$2, and a \$1 change in the stock price causes a 50% change in the call option price.

QUESTION 21.2: DELTA AT INFINITY

European call and put options are written on a share of stock, with a *strike price* of $80. As the *stock price* approaches infinity, which of the following are correct?

A. The call option value approaches 1, the call option delta approaches 1, the put option value approaches 0, and the put option delta approaches –1.
B. The call option value approaches infinity, the call option delta approaches 1, the put option value approaches 0, and the put option delta approaches –1.
C. The call option value approaches infinity, the call option delta approaches infinity, the put option value approaches 0, and the put option delta approaches 0.
D. The call option value approaches infinity, the call option delta approaches 1, the put option value approaches 0, and the put option delta approaches 0.
E. The call option value approaches 1, the call option delta approaches 1, the put option value approaches 0, and the put option delta approaches 0.

Answer 21.2: D

The *intrinsic value* of the call option is the stock price minus the strike price. The market price of the call option exceeds this, since the option also has a time-element value. As the stock price approaches infinity, the intrinsic value of the call option approaches infinity, so the full value of the call option approaches infinity.

As the stock price increases to infinity, the investor surely exercises the call option. A dollar rise in the stock price increases the value of the call option by one dollar, so the delta becomes unity.

The put option is exercised only if the stock price is less than the strike price at the expiration date. As the stock price approaches infinity, the probability that the put option will be exercised approaches zero, and its value approaches zero.

The delta of the put option is the change in price of the put option divided by the change in price of the underlying security. As the stock price approaches infinity, the value of the put option approaches zero. For a one dollar change in the stock price, the change in the value of the put option approaches 0 – 0 = 0.

Question 21.3: Risk-Neutral Valuation

In the binomial tree models that are used to value options, each node leads to two future nodes, an *up* path and a *down* path. All but which of the following statements is true?

A. As the true probability of an upward movement in the stock price increases, the value of a call option on the stock increases and the value of a put option on the stock decreases.
B. The true probabilities of future up or down movements are already incorporated into the price of the stock and don't affect the prices of options on the stock.
C. In a risk-neutral world, investors require no compensation for risk, and the expected return on all securities is the risk-free interest rate.
D. If the up movement is +20%, the down movement is –20%, and the expected return on a stock is 12%, the true probability of an up movement is p × 20% + (1 – p) × (–20%) = 12% ⇛ p = (12% – (–20%)) / (20% – (–20%)) = 32% / 40% = 80%.
E. If the up movement is +20%, the down movement is –20%, and the risk-free rate is 5%, the true probability of an up movement is p × 20% + (1 – p) × (–20%) = 5% ⇛ p = (5% – (–20%)) / (20% – (–20%)) = 25% / 40% = 62.5%.

Answer 21.3: A

This illustrative test question reviews the crux of the binomial tree pricing method. We explain each statement by examples.

TWO OPTIONS

Statement A above sounds reasonable. To see why it is not correct, we examine the two options here.

Stock A is trading at $20 a share. Option A is a European call option on Stock A with a strike price of $21. The stock price in three months will be either $22 or $18. Stock A has a *90%* chance of moving up to $22 and a *10%* chance of moving down to $18. This is the option with a high probability of an upward movement in the stock price.

Stock B is trading at $20 a share. Option B is a European call option on Stock B with a strike price of $21. The stock price in three months will be either $22 or $18. Stock B has a *75%* chance of moving up to $22 and a *25%* chance of moving down to $18. This is the option with a lower probability of an upward movement in the stock price.

Stocks A and B are unrelated. Stock A may move up when Stock B moves down, and Stock B may move up when Stock A moves down.

A. Which option has the higher expected value *at the expiration* date, option A or B?
B. Which option is worth more *now*, option A or option B?

C. Which option is *riskier*, option A or option B? By *riskier*, we mean greater systematic risk.

Solution 21.3:

Option pricing is tricky. In the dialogue between Jacob and Rachel, Jacob's comments are not all correct, though they make sense at first.

Jacob: For Part A, we work out the expected values of Option A and Option B at the expiration date:

- *Option A:* In three months, Stock A will be either $22 or $18. If Stock A is worth $22, the call option is worth $22 – $21 = $1. If Stock A is worth $18, the call option is worth $0. The probability that Stock A will be worth $22 is 90%, so the *expected value* of Option A in three months is 90% × $1 + 10% × $0 = $0.90.
- *Option B:* In three months, Stock B will be either $22 or $18. If Stock B is worth $22, the call option is worth $22 – $21 = $1. If Stock B is worth $18, the call option is worth $0. The probability that Stock B will be worth $22 is 75%, so the *expected value* of Option B in three months is 75% × $1 + 25% × $0 = $0.75.

Option A has the higher expected value in three months.

Rachel: That's correct.

Jacob: The difference between Stocks A and B is the probability of increasing in value. We know nothing else about the stocks or the options, so if Option A is worth more at the expiration date, it is worth more now.

For the risk of each option, we examine the standard deviation of their prices; higher standard deviation means higher risk.

Option A: The *mean* value of option A in three months is $0.90. The *variance* of the option value in three months is

$$90\% \times (\$1.00 – \$0.90)^2 + 10\% \times (\$0.00 – \$0.90)^2 = \$0.09$$

The *standard deviation* in three months is $0.09^{0.5}$ = $0.30.

Option B: The *mean* value of option B in three months is $0.75. The *variance* of the option value in three months is

$$75\% \times (\$1.00 – \$0.75)^2 + 25\% \times (\$0.00 – \$0.75)^2 = \$0.19$$

The *standard deviation* in three months is $0.19^{0.5}$ = $0.43

We infer that Option B is the riskier option.

Rachel: The arithmetic is correct. Option A has the higher expected value in three months. Option B has the higher variance and the higher standard deviation of its expected value.

But the two options have the *same present value*, and *Option A* is the riskier option.

Option A and Option B have the same present value because the pricing of the option does *not* depend on the *actual* probabilities of the stock movements. The actual probabilities are referred to as the *subjective* probabilities, *actual* probabilities, or *real world probabilities*, to distinguish them from *risk-neutral probabilities*.

To price these options with the binomial tree pricing method, we need the risk-free interest rate. We are not told this rate, but it is the same for both stocks. The risk-free interest rate does not depend on the particular option that we are pricing. Since everything else in the practice problem is the same for the two options, they must have the same price.

(Jacob is not yet persuaded; we come back to this topic further below.)

Risk and Capitalization Rates

The standard deviation and the variance are not necessarily good proxies for risk. We use only *systematic* risk to price securities, not the total standard deviation or variance.

We determine the *relative* risk of the two options by comparing their capitalization rates. The option with the higher capitalization rate has the greater risk.

For any problem comparing the risk of securities, security #1 must have the higher variance but security #2 may have the higher systematic risk. In fact, security #1 may have a very high variance and standard deviation and security #2 may be risk free, but if security #1 is inversely correlated with the overall market return, security #2 is riskier.

Jacob: How can a security have less risk than a security that is risk-free?

Rachel: A security which has negative risk is less risky. A security which has a negative beta has negative risk.

We *don't know the risk of either option*. We know only that they have the same present value and that Option A has the higher expected value in three months. This implies that Option A has the higher capitalization rate and is the riskier security.

Exercise 21.4: Two Stocks

Jacob: I recognize the phrases you used; they come from the Brealey and Myers text. The text discusses total risk and systematic risk, along with total variance (or total standard deviation) vs the CAPM beta. But I don't understand how these options can have the same value. You admit that the options have different expected values in three months. That is all we know about these two options; you admit that we don't know the risk of the options. If all we know about two securities is that one has a greater expected value in three months, shouldn't we assume it have the greater current value as well?

Rachel: Let us revise the illustration to speak about the stocks, not about call options.

Stock A is trading at $20 a share. Option A is a European call option on Stock A with a strike price of $21. The stock price in three months will be either $22 or $18. Stock A has a 90% chance of moving up to $22 and a 10% chance of moving down to $18.

Stock B is trading at $20 a share. Option B is a European call option on Stock B with a strike price of $21. The stock price in three months will be either $22 or $18. Stock B has a 75% chance of moving up to $22 and a 25% chance of moving down to $18.

A. Which stock has the higher expected value in three months' time, stock A or B?
B. Which stock is worth more *now*, stock A or stock B?
C. Which stock is *riskier*, stock A or Stock B?

Solution 21.4:

Part A: Expected Stock Value

We work out the expected values of Stock A and Stock B.

- Stock A has a 90% probability of being worth $22 in three months and a 10% probability of being worth $18 in three months. The expected value of Stock A in three months is 90% × $22 + 10% × $18 = $21.60.
- Stock B has a 75% probability of being worth $22 in three months and a 25% probability of being worth $18 in three months. The expected value of Stock A in three months is 75% × $22 + 25% × $18 = $21.00.

Option A has the higher expected value in three months.

Part B: The present value of the stocks is $20 for each stock (as mentioned in the problem). The present value of a stock is its market value, which is $20 for both stocks.

Risk Capitalization Rates

Part C: Whichever stock has the higher capitalization rate is the riskier stock.

- *Stock A:* The capitalization rate for a three month period is \$21.60 / \$20.00 = 8.00%. The effective annual rate is 1.08^4 = 36.05%.
- *Stock B:* The capitalization rate for a three month period is \$21.00 / \$20.00 = 5.00%. The effective annual rate is 1.05^4 = 21.55%.

Stock A is the riskier stock.

Jacob: How can stock A be the riskier stock? The only difference between Stock A and Stock B is that Stock A has a higher probability of increasing in value. Stock B has the higher probability of decreasing in value. Shouldn't this make Stock B the riskier stock?

Rachel: Stocks A and B are not related. Perhaps stock A is a high beta stock, which increases in value when the overall stock market increases in value. Stock B might be a low beta (defensive) stock, which increases in value when the overall stock market declines and decreases in value when the overall stock market increases.

For the pricing of options, we are not concerned with the type of risks accompanying the underlying securities. For whatever reason, Stock A is the riskier stock, as shown by the market valuation of the stock. The option contract has the same risk as the underlying security, but the risk is *leveraged* (magnified). Option A has the higher expected value at the expiration date, but the same present value, so it is the riskier security.

States of the World

Jacob: We have two scenarios, a favorable scenario and an adverse scenario, or a favorable state of the world and an adverse state of the world.

- Stock A has a 90% probability of being in the favorable state.
- Stock B has a 75% probability of being in the favorable state.

How can you say that Stock A is not worth more than Stock B, if it is always in at least as good a state of the world and sometimes in a better state of the world?

Rachel: You are mis-using the terms. A *state of the world* is just what it says: all securities have the same probability of being in that state, since the state has nothing to do with the security. If there were just two states of the world, and Stock A had a 90% probability of being in one of these states in three months time, Stock B would have the same probability of being in that state, since the entire world is in that state.

In the problem with Stocks A and B, there are at least thee states of the world, and there may be many more. The two stocks don't move up or down together.

Let us show the three possible states:

State	*Probability*	*Stock A*	*Stock B*
α	75%	up	up
β	15%	up	down
γ	10%	down	down

Valuation of Securities

Let us return to the original illustrative test question.

Rachel: It is natural to assume that as the probability of an upward movement in the stock price increases, the value of a call option on the stock increases and the value of a put option on the stock decreases. This is not the case.

Jacob: if there is a greater chance for the stock price to increase, then the option must be worth more.

Rachel: We do not value the option in absolute terms. We calculate its value in terms of the price of the underlying stock. The probabilities of future up or down movements are already incorporated into the price of the stock. We do not need to take them into account again when valuing the option in terms of the stock price.

Jacob: When we price an option, we get a dollar value, not a factor that is applied to the stock price. What do you mean by *in terms of the price of the underlying stock*?

Rachel: The value of the option depends on the value of the underlying security, not just on the future cash flows of the derivative security.

We summarize the conclusion regarding stocks A and B as follows:

> The current stock price is $20. The stock price at the end of the period is either $22 or $18. We are pricing a call option on the stock with an exercise price of $21. The price of the call option does ***not*** depend on the actual probabilities of moving up to $22 or down to $18.

The Black-Scholes formula prices an option in terms of five input values: the stock price, the exercise price, the expected, the risk-free rate, and the stock price volatility. Let us denote the exercise price as a multiple of the stock price, such as 90% of the stock price, or 115% of the stock price. If we double the stock price, the call option value and the put option value both double. If we multiply the stock price by a constant *k*, the values of the call option and the put option are multiplied by *k*.

- In d_1 and d_2, the stock price and the exercise price appear only in the ratio of one to the other; multiplying them both by the same constant doesn't change the ratio.
- In the formulas for the call and put values, the constant k factors out.

Constraints

Jacob: The *up* and *down* probabilities in the binomial tree pricing model are not related to the real-world probabilities of *up* and *down* movements in the stock price. Can we use any *up* and *down* movements that we wish?

Rachel: No. Two constraints limit the choice. The no-arbitrage constraint requires that the risk-free interest rate lie between the up and down movements.

Question 21.5:

The risk-free interest rate is 10% per annum. We are pricing a one year option with a four period binomial tree. Which of the following are possible values for the up and down movements of the binomial tree?

A. Up = +10%; down = +5%.
B. Up = +3%; down = +2%.
C. Up = +2%; down = –2%.
D. Up = +2%; down = –0%.
E. Up = 0%; down = –2%.

Answer 21.5: B

The up and down movements must straddle the risk-free rate: the up movement must be greater and the down movement must be lower. This question uses a four period binomial tree, so each period is three months, or one quarter of a year. The risk-free rate for three months is $1.10^{1/4} - 1 = 2.41\%$ if we use annual compounding and $10\% \times 1/4 = 2.5\%$ if we use continuous compounding.

Jacob: Why do we often use continuous compounding for options pricing? Does this make a difference?

Rachel: No. Continuously compounded rates are easier for the Black-Scholes model and for pricing with various different terms to maturity. 10% per annum with continuous compounding is 2.5% per quarter with continuous compounding. We could use annual effective yields and take the fourth root.

Jacob: What should we use for the final exam?

Rachel: The final exam problems use annual effective yields. If a final exam problem uses continuous compounding, it explicitly says that the effective interest rate is e^r.

In this problem, the risk-free interest rate for a three month period is 2.4% with annual compounding and +2.5% with continuous compounding. All statements except for statement B lead to arbitrage profits.

- If the stock price followed the movements in statement A, we would *short the risk-free bond* and *buy the stock*. We pay 2.4% or 2.5% each quarter of interest payments, and we *earn at least* 5% each quarter on the stock.

- If the stock price followed the movements in statement 3, we would *short the stock* and *buy the risk-free bond*. We *pay at most* 2.0% each quarter for the stock, and we earn at least 2.4% each quarter on the bond.

Jacob: What does it mean to *short the bond*?

Rachel: To *short the bond* means to borrow at 10% per annum.

The arbitrage constraint is that $u > r > d$. r must be positive, so u must also be positive. d can be positive or negative. For a high volatility stock, u is very high and d is usually negative. For a low volatility stock, u is slightly above the risk-free rate and d is slightly below the risk-free rate.

Jacob: Why must r be positive?

Rachel: If r were negative we would have an arbitrage opportunity, by shorting the risk-free bond and placing the cash under the mattress.

Question 21.6:

The risk-free interest rate is 10% per annum with continuous compounding. We are pricing a one year option with a four period binomial tree. The volatility parameter σ of the stock is 30% per annum. Which of the following are possible values for the up and down movements of the binomial tree? All the movements below use continuous compounding.

1. Up = +10%; down = –10%.
2. Up = +30%; down = –30%.
3. Up = +15%; down = –15%.

A. 1 and 2 only.
B. 1 and 3 only.
C. 2 and 3 only.
D. 1, 2, and 3.
E. None of A, B, C, or D is correct.

Answer 21.6: Volatility Constraint

E (only Statement 3 is correct)

The most common method of selecting the up and down movements is to set

$$i = e^{\sigma \times \sqrt{t}} \text{ and } d = e^{-\sigma \times \sqrt{t}}$$

In this illustrative test question, t = ¼ and σ = 30%, so $e^{\sigma \times \sqrt{t}} = e^{15\%}$ and $e^{-\sigma \times \sqrt{t}} = e^{-15\%}$. The textbook mentions this method. It is not the only method, and it does not always give an arbitrage free scenario, but it is the easiest method to use (when it works).

Jacob: If we used the movements in statement 1 or statement 2, would the computed option price be too high or too low?

Rachel: The movements in statement 1 imply an annual volatility of 20%. Since the actual volatility is 30% per annum, the option price computed from the binomial tree pricing method would be too *low*. This is true for both a call option and a put option.

The movements in statement 3 imply an annual volatility of 40%. Since the actual volatility is 30% per annum, the option price computed from the binomial tree pricing method would be too *high*. This is true for both a call option and a put option.

Jacob: Are there other possibilities for the up and down movements?

Rachel: There are many possibilities. We could use +20% and –10%, and we could use +10% and –20%. Both of these give a quarterly volatility of about +15%, which is the same as an annual volatility of 30%.

Jacob: On the final exam, what should we use?

Answer: If you are given the stock price volatility but not a pair of price movements, use

$$u = e^{\sigma \times \sqrt{t}} \text{ and } d = e^{-\sigma \times \sqrt{t}}.$$

Question 21.7: *Model Components*

To value an option, we use a binomial tree where the risk-free interest rate, r, the up movement, u, the down movement, d, and the time step Δt are kept constant. Which of the following statements are correct?

1. The option delta, Δ, changes over time.
2. The risk-neutral probability, p, changes over time.
3. To maintain a riskless hedge using an option and the underlying stock, we must adjust our holdings in the stock periodically.

A. 1 and 2 only.
B. 1 and 3 only.
C. 2 and 3 only.
D. 1, 2, and 3.
E. None of A, B, C, or D is correct.

Rebalancing

Answer 21.7: "B: 1 and 3 only."

The binomial tree pricing method is a discrete method. To keep a risk-free portfolio, we re-balance the portfolio at every step of the binomial tree model. The Black-Scholes model is a *continuous* model. To keep a risk-free portfolio, we re-balance the portfolio continuously in the Black-Scholes pricing model.

Jacob: Does an investor using Black-Scholes have to hold a risk-free portfolio?

Rachel: No. The risk-free portfolio is used for the pricing (arbitrage) argument. As long as some investors seek out mispriced securities and form risk-free portfolios to earn arbitrage profits, the derivative securities will not remain mispriced in an efficient market.

Jacob: Are there actually any arbitragers who seek out mispriced derivative securities?

Rachel: Yes; some hedge funds seek mispriced derivative securities to gain arbitrage profits. The hedge fund searches for *relative* mispricing. It can't say that option A is overpriced or option B is underpriced, since the price depends on the volatility of the underlying security, which is hard to estimate. But the hedge fund can say that *a certain call option* on a stock is *overpriced relative* to a certain put option on the same stock. The hedge fund would buy the put option and simultaneously sell the call option.

Jacob: Is it easy to find such pairs of relatively mispriced derivative securities?

Rachel: It is hard to find them in efficient markets, such as the markets for derivative securities on stock indices. It is easier to find them in thinly traded markets for unusual options, such as the markets for bonds with embedded options.

Jacob: Rebalancing incurs transaction costs. Do traders form risk-free portfolios and rebalance them?

Rachel: Yes. Some financial institutions who sell options rebalance their portfolios every day.

Risk-Neutral Probability

Statement 2 is correct. We assume that the risk-free rate is not stochastic, and that the values of the *up* and *down* parameters remain constant.

The risk-neutral probability p satisfies $p \times u + (1 - p) \times d = r$. That's what risk-neutral valuation means: if investors are risk-neutral, they are indifferent between the uncertain return with the *up* and *down* parameters or the certain return at the risk-free rate.

Option Delta

The delta changes at the each node of the tree. The delta depends on the price of the stock (among other variables). The option delta is the ratio of the price change of the call option to the price change of the stock itself.

- If the price of the stock increases, the value of a call option becomes more closely correlated with the price of the stock.
- If the price of the stock decreases, then the value of a call option becomes less closely correlated with the price of the stock.

If the price of the stock increases to infinity, delta becomes unity.

At each node, the stock price of the stock moves up or down and the option delta changes.

Jacob: Is this the only reason for the change in the option delta? If the stock price does not change, would the option delta change?

Rachel: Yes. The option delta changes with the passage of time. Suppose we have a one year call option on stock ABC with strike price $85 and stock price $80. Because the duration of the option is long (one year), the call option may be worth now $10. If the stock price increases from $80 to $81, the call option value may increase from $10 to $10.50, for a 50% delta.

Suppose the stock price is $80 one day before the expiration of the option. The value of the option is a few cents. If the stock price increases by $1 during the morning, the value of the option may increase by five cents, for a 5% delta.

Jacob: What if the stock were far in the money instead of far out of the money? If the stock were $90 one day before the expiration of the option, what would the delta be?

Rachel: At a stock price of $90, the option may be worth $5 plus four cents. If the stock price rises to $91, the option may be worth $6 plus two cents. The option delta is 98%.

The delta represents the amount of shares that one must sell short to hedge one long option, or the amount of shares that one must hold long to hedge against one short option. If the delta changes, then the hedging portfolio must be re-balanced.

EXERCISE 21.8: *EUROPEAN CALL OPTION*

A stock price is currently \$40. At the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. A one-month European call option is trading with a strike price of \$39?

A. What is the risk-free portfolio of 1 call option ± some shares of stock?
B. What is the option delta derived by solving for the risk-free portfolio?
C. What is the option delta derived by the ratio of the change in the option value to the change in the stock price?
D. What is the return on the risk-free portfolio?
E. From the returns in the two scenarios (the two states of the world), what is the present value of the call option?
F. What is the risk-neutral probability of a rise in the stock price?
G. What is the expected value of the call option at expiration in a risk-neutral world?
H. What is the present value of the call option?

Solution 21.8:

Two mathematically equivalent methods of solving these problems are the no-arbitrage method (the option delta method) and the risk-neutral valuation method.

The no-arbitrage method lets you see what is happening; the risk-neutral valuation method is faster for solving problems. In some problems, it may not be clear how to use the risk-neutral valuation method. The no-arbitrage method shows the intuition; once you have solved the problem, you can see how to use the risk-neutral valuation method to solve it more quickly.

Part A: The no-arbitrage argument runs as follows. We set up a portfolio consisting of the call option *c* minus *delta* (Δ) shares of stock such that this portfolio is risk free over the time interval in the problem: $c - \Delta S$

Part B: At the end of the month, we know the value of the option in two scenarios. The strike price is \$39.

- If the stock price increases to \$42, then the call option is worth \$3.
- If the stock price decreases to \$38, then the call option expires worthless.

If the portfolio is risk-free, its value in these two scenarios must be the same:

$$\$3 - \Delta \times \$42 = \$0 - \Delta \times \$38, \text{ or}$$
$$\Delta = 3/4.$$

Part C: We wrote out the intuition to make the procedure clear. More simply, the delta is the ratio of the change in the option price to the change in the stock price:

$$\Delta = (\$3 - \$0) / (\$42 - \$38) = 3/4.$$

Part D: If the portfolio is risk-free, then its return must be the risk-free rate. This is true by a no-arbitrage argument.

- If its return were higher than the risk-free return, then we could buy the portfolio and sell short risk-free bonds to pay the purchase price. This would give us a positive return for an investment of $0, which is an example of arbitrage.
- If the return on the portfolio were less than the risk-free return, we could sell the portfolio short and purchase risk-free bonds, again making a risk-free profit.

The risk-free rate is 8% with continuous compounding. The risk-free return for one month is $e^{8\%/12} = e^{0.00667} = 1.0067$.

Jacob: For the final exam, will any problems use continuous compounding?

Rachel: Most problems use effective annual yields. If a problem uses another compounding interval, it will explicitly tell you.

Part E: The one month return on the risk-free portfolio must be +0.67% in the two scenarios. Let's use the *up* scenario to determine the option price.

In the *up* scenario, the stock begins at $40 and ends at $42. The option price begins at *c* and ends at $3. The equation is

$$(c - 3/4 \times \$40) \times 1.0067 = (\$3 - 3/4 \times \$42), \text{ or}$$
$$c = \$1.69$$

We do the same with the *down* scenario where the stock begins at $40 and ends at $38 and the option price begins at *c* and ends at $0:

$$(c - 3/4 \times \$40) \times 1.0067 = (\$0 - 3/4 \times \$38), \text{ or}$$
$$c = \$1.69$$

Part F: The risk-neutral valuation method using the binomial tree puts this algebraic procedure into formulas. First we find the probability of the *up* movement *in a risk-neutral world*. If the risk-free rate is 8% per annum with continuous compounding, the equation becomes

$$p \times u + (1 - p) \times d = e^{rT}$$
$$p = (e^{rT} - d) / (u - d)$$

In our problem, u = 42/40 = 1.05 and d = 38/40 = 0.95, so we have

$$p \times 42/40 + (1 - p) \times 38/40 = e^{8\%/12}$$
$$p = (1.0067 - 0.95) / (1.05 - 0.95) = 0.567$$

Part G: We now value the option using risk-neutral valuation. The values of the option at the end of the month are \$3 in the "up" state and \$0 in the "down" state. The expected value of the option at the expiration date in the risk-neutral world is

Part H: The value of the option at the beginning of the month is

$$c = 1/1.0067 \times [0.567 \times \$3 + (1 - 0.567) \times \$0] = 1.689$$

This is the same answer as we got in the no-arbitrage method.

Cookbook

We have emphasized the intuition; that's what you need to understand the text. We also provide a seven-step cookbook procedure for exam problems.

1. Use the volatility of the stock price to determine the up and down movements: the up movement is $e^{\sigma\sqrt{t}}$ and the down movement is $e^{-\sigma\sqrt{t}}$.

Jacob: This means that d = 1/u. But some examples don't have this relation.

Rachel: There are many ways to choose the up and down movements. The movements must provide a volatility of σ. The expressions above do this; other choices do this as well.

Jacob: Don't we also have to take into consideration the stock's expected return μ? After all, if the stock has an expected return of +20% and we choose up and down movements of +15% and –5%, the binomial tree won't agree with the actual movements of the stock.

Rachel: The binomial tree pricing method assumes a risk-neutral world. In a risk-neutral world, the stock's expected return is the risk-free interest rate. We need the relation that

down movement < risk-free rate < up movement.

We don't need to do anything with the actual drift of the stock (μ).

2. Form the recombining binomial tree. From each node in column *i*, we use the up movement and the down movement to get the nodes in column *i+1*.

Jacob: Is the binomial tree always recombining?

Rachel: If we use this definition of the up movement and the down movement the binomial tree is recombining. If is also recombining if we have a proportional stock dividend.

Jacob: When is the tree not recombining?

Rachel: The binomial tree is not recombining when there is a constant dollar dividend. {Note: The final exam problems do not have dividend paying stocks.}

3. Determine the risk-neutral probability of the up movement. The formula for the risk-neutral probability of the up movement is

$$p = (e^{rT} – d) / (u – d)$$

Some authors call this the *equivalent Martingale probability*; this term is not in the syllabus readings, but you will see it on the Course 6 readings (Panjer's textbook).

4. Determine the value of the derivative (the option) in the final column of nodes. The final column is the expiration date of the derivative, so the value of the derivative is the intrinsic value; there is no time value anymore to the derivative.

Jacob: Are these values related to the risk-neutral probabilities in the previous step?

Rachel: No. These values have not included the probability of arriving at any of the nodes in the final column. You can reverse these two steps; they are not related.

5. Determine the expected value of the derivative at the final column of nodes using the risk-neutral probabilities of reaching each node. This gives us the expected value of the derivative in a risk-neutral world.

This is *not* the expected value of the derivative at the expiration date. If we obtained the expected value of the derivative at the expiration date, we would not be able to go on, since we don't know the proper capitalization rate to discount back to the present time. Rather, this is the expected value *in a risk-neutral world*.

6. Discount this future expected value to the present time using the risk-free interest rate. Using the risk-neutral probabilities to go forward in time and the risk-free interest rate to discount back in time exactly offset each other. The present value of the derivative that we obtain with this method is true both for a risk-neutral world and for the real world. In fact, there is no such thing as a risk-neutral world; this is just a nice term that helps us think about this procedure. This is simply the present value of the derivative in the real world.

Exercise 21.9: *Optionese*

Explain the no-arbitrage (option delta) and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

No-arbitrage

The no-arbitrage argument says that if we have only 2 paths leading from each node, then we can set up a risk-free portfolio consisting of one option and the short sale of delta shares of the underlying asset. We know that we can set up such a risk-free portfolio since we have one variable (delta) and one equation (the value of the risk-free portfolio must be equal in the two states stemming from this node).

Once we set up the risk-free portfolio, we know that the return on the risk-free portfolio must be equal to the risk-free rate. We set up an equation saying that the value of the portfolio at the beginning of the period times the risk-free interest rate equals the value of the portfolio (at either node, since they are equal) at the end of the period. This equation solves for the value of the option.

Risk Neutral Valuation

The risk-neutral valuation approach stems from the no-arbitrage approach. The risk-neutral valuation approach says that our previous argument was not dependent upon the risk aversion of investors.

Jacob: We never discuss the risk aversion of investors in financial theory, do we?

Rachel: We do this all the time, though it's not obvious. We discuss the returns of various securities, such as the returns on stocks, or corporate bonds, or mortgages, or real estate. These securities have different returns because they have different types of risk and because investors are risk averse. Any time that we use the actual return of a security in an argument, we are using the risk aversion of investors.

In the no-arbitrage argument, we never discuss the actual return on the underlying security, so we never use the risk aversion of investors for our argument. If so, the same argument is true in any type of world, whether a risk-neutral world or a world where investors are risk averse. It's simplest to solve for the value of the option in a risk-neutral world. We solve for the value of the option in that type of world, but this value is the same in any type of world.

EXERCISE 21.10: *EUROPEAN PUT OPTION*

A stock price is currently \$50. It is known that at the end of six months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. A six-month European put option has a strike price of \$50.

A. What are the up and down movements in the stock price?
B. What is the value of the put option if the stock price moves up or down?
C. What is the risk-neutral probability of a rise in the stock price?
D. What is the expected value of the put option at expiration in a risk-neutral world?
E. What is the present value of the put option?

Solution 21.10: Risk-neutral Probability

Part A: To determine the risk-neutral probability, we must know the values of the *up* and *down* movements and the risk-free interest rate. We do not have to know the current stock price or the exercise price, since the risk-neutral probability doesn't change as long as the *up* and *down* movements and the risk-free interest rate don't change.

Jacob: What if we are given the stock values at the nodes, not the up and down movements?

Rachel: Call the stock values S_0, S^+, and S^-. The up movement is S^+ / S_0. The down movement is S^- / S_0.

In our problem, $e^{rT} = e^{10\%/2} = 1.0513$, $u = 55/50 = 1.10$, and $d = 45/50 = 0.90$.

Jacob: Is it true that u > 1 and d < 1?

Rachel: No. All we need is that $u > e^{rt}$ and $d < e^{rt}$. If the risk-free interest rate is 8% for the time period, then u = 6% and d = –5% doesn't work, but u = +12% and d = +1% works fine.

Part B: If the stock price moves up to 55, the put option expires worthless (= \$0). If the stock price moves down to 45, the put option is worth 50 – 45 = 5.

Part C: The formula for the risk-neutral probability is $p = (e^{rT} – d) / (u – d) =$

$$p = (1.0513 – 0.90) / (1.10 – 0.90) = 0.756$$

Part D: The expected value of the put option at expiration in the risk-neutral world is

$$[0.756 \times \$0 + (1 – 0.756) \times \$5] = \$1.22$$

Part E: The present value of the put option is

$$1/1.0503 \times [0.756 \times \$0 + (1 - 0.756) \times \$5] = 1.1588.$$

EXERCISE 21.11: TWO PERIOD BINOMIAL MODEL

A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. A one-year European call option has a strike price of \$100.

A. Draw the binomial tree, using the following nodes: Node A is the initial stock price, node B is the stock price after one "up" movement, node C is the stock price after one "down" movement, node D is the price after two "up" movements, node E is the price after one "up" movement followed by one "down" movement (or one "down" movement followed by one "up" movement), and node F is the price after two "down" movements. Give each node two figures: the top number is the price of the underlying asset (such as the stock) and the bottom figure is the price of the option at that moment in time. Fill in the values for nodes D, E, and F.
B. What is the risk-neutral probability of a rise in the stock price?
C. What is the value of the option at node B?
D. What is the value of the option at node C?
E. What is the value of the option at node A?
F. Find the value of the option at node A using a one-line formula.

The first several times that you do this, draw the binomial tree. Almost all of the examples in the textbook have recombining trees. This means that the price of the underlying security is the same whether an *up* movement is followed by a *down* movement or a *down* movement is followed by an *up* movement. For options on non-dividend paying stocks, where the *up* and *down* movements don't change over time, this means that the *up* and *down* movements are multiplicative factors.

Practice drawing the binomial tree for several problems. The formats for both one period trees and two period trees are shown in the textbook.

Nodes D, E, and F are at the exercise date. Fill in the values of the options at the exercise date based on the exercise price, the price of the underlying asset, and the type of option (call or put). That's as far as we can get without any options pricing technique. For nodes A, B, and C, we must use either the no-arbitrage method or the risk-neutral valuation method.

Using the figures in this exercise, we first fill in the stock prices. We start at \$100, and we move up or down by 10% in each period. We calculate *uu* as +21%, *ud* as –1%, and *dd* as –19%, so the ending stock prices are \$121, \$99, and \$81 in nodes D, E, and F.

This is a European call option, so the option values at exercise are \$21, \$0, and \$0 at nodes D, E, and F, respectively.

Jacob: How does the option pricing technique differ for European call vs put options?

Rachel: Only one piece differs:

- The prices of the underlying asset at each node do not depend on the type of option.
- The risk-neutral probability does not depend on the type of option.
- The ending values of the option (at the exercise date) do depend on the type of option.
- The rest of the risk-neutral valuation procedure does not depend on the type of option.

Part B: The risk-neutral probability does not vary over the life of the option, since the *up* and *down* movements do not change and the risk-free interest rate does not change. We have

$$p = (e^{rT} – d) / (u – d)$$

$e^{rT} = e^{8\%/2} = 1.0408$, $u = 1.10$, and $d = 0.90$, so

$$p = (1.0408 – 0.90) / (1.10 – 0.90) = 0.704$$

Part C: T is the length of the period, which is six months or ½ of a year in this problem. We use the risk-neutral valuation procedure to solve for the value of the option at each node, using $e^{-rT} = 1/1.0408 = 0.9608$. At node B, we have

$$\text{value of call at node B} = 0.9608 \times [0.704 \times \$21 + (1 – 0.704) \times \$0] = \$14.2054.$$

Part D: For node C, the arithmetic is easy. Since the option value is $0 at both node E and node F, the option value is $0 at node C as well. Conceptually, this says that if the option will be worth $0 at the next time period, whether an "up" or "down" movement occurs, then it is worth $0 now. This is a characteristic of the binomial tree model. In real life, it is almost always possible that the option will end up worth something next period, either by a large rise in the stock price (for a call option) or by a large decline in the stock price (for a put option).

Part E: For node A, the value of the call option is

$$0.9608 \times [0.704 \times \$14.2054 + (1 – 0.704) \times \$0] = \$9.6092.$$

Part F: The one-line formula for the value of the option is

$$f = e^{–2r\Delta t} [p^2 f_{uu} + 2p(1–p) f_{ud} + (1–p)^2 f_{dd}]$$

$$f = e^{–2(0.08)(0.5)}[(0.704^2)(\$21) + 2(0.704)(1–0.704)(\$0) + (1–0.704)^2(\$0)] = \$9.6092$$

EXERCISE 21.12: TWO PERIOD PUT

A. For the previous exercise, what is the value of a one-year European put option with a strike price of $100?
B. Verify that the European call and the European put prices satisfy put-call parity.

Part A: This question helps you see the differences between call option pricing and put option pricing. The only difference is the values of the option at each node in the final column (i.e., at the exercise date). For the put option, the values of the option are $0 at node D (up-up), $1 at node E (up-down), and $19 at node F (down-down). We use these numbers to get

$$f = e^{-2(0.08)(0.5)}[(0.704^2)(\$0) + 2(0.704)(1–0.704)(\$1) + (1–0.704)^2(\$19)] = \$1.9208.$$

Part B: The put-call parity relationship helps verify the answer. The put-call parity relationship says that "call plus cash = put plus stock." Cash is the amount of cash that would accumulate to the strike price if invested at the risk-free interest rate. Similarly, the other values in the put call parity relationship are the values at the initial date.

In this problem, cash = $e^{-2(0.08)(0.5)}$($100) = $923.12. The put-call parity relationship says that $9.6092 + $92.31 = $1.9208 + $100. This is true.

RISKLESS PORTFOLIO

Exercise 21.13: Consider the situation where stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

Varying Delta

The volatility of the stock price doesn't change over the life of the option in the binomial tree model. However, the price of the underlying asset changes and the time remaining until the exercise date changes, and both of these changes affect the option delta.

The relative amounts of stock shares versus options in the risk-free portfolio at any time depend on the option delta at that moment. Thus, the relative amounts change over the life of the option. To keep the portfolio risk-free, the portfolio must be rebalanced at each node of the binomial tree.

EUROPEAN CALL OPTION

Exercise 21.14: A stock price is currently \$50. It is known that at the end of two months it will be either \$53 or \$48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of \$49?

Solution Methods

The textbook often asks you to use no-arbitrage arguments in the solution. We used no-arbitrage arguments in our solution to Question 9.1. These arguments take a long time to write down, and they are conceptually the same for all the problems. The model solutions here simply show the answers from the risk-neutral valuation method. In your mind, make sure that you can formulate the no-arbitrage arguments for each situation. On the exam, you may be asked to evaluate the option delta at various nodes.

The risk-free portfolio changes at each node. For a two-step binomial tree, we must form as many as 3 risk-free portfolios for the no-arbitrage argument.

This problem is a simple one step binomial tree. The “up” movement is 53/50 = 1.06. The down movement is 48/50 = 0.96. For the risk-neutral probability we have

$$p = (e^{rT} - d) / (u - d)$$

Note that the “up” movement does not have to be the reciprocal of the “down” movement. In fact, the examples in Hull’s text generally do not use reciprocals; that is, $u \neq 1/d$. However, Hull notes that certain options pricing formulas do use reciprocal movements, where $u = 1/d$. On the exam, you might be expected to assume that $u = 1/d$.

In this problem, $e^{rT} = e^{10\%/6} = 1.0168$, so

$$p = (1.0168 - 0.96) / (1.06 - 0.96) = 0.568$$

We take the expected value of the “up” and “down” values of the call option at the risk-neutral probability and we discount by the risk-free interest rate to get

$$\text{value of call} = 1/1.0168 \times [0.568 \times \$4 + (1 - 0.568) \times \$0] = \$2.2347$$

Remember that the value of the call option at the exercise date is Max $(0, S_T - X)$. Review the model solution to Question 9.1 and fill in the no-arbitrage argument.

TWO PERIOD PUT

Exercise 21.15: A stock price is currently \$80. It is known that at the end of two months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a four-month European put option with a strike price of \$80?

Option Type

This question is just like the previous one, except that the call option is replaced by a put option and the parameters are changed. Since the type of option is relevant only for the boundary conditions, the changes to the solution are trivial.

This problem is a one step binomial tree. The “up” movement is 85/80 = 1.0625. The down movement is 75/80 = 0.9375. We convert these to “up” and “down” movements.

For the risk-neutral probability we have

$$p = (e^{rT} – d) / (u – d)$$

In this problem, $e^{rT} = e^{5\%/3} = 1.0168$, so

$$p = (1.0168 – 0.9375) / (1.0625 – 0.9375) = 0.63445$$

We take the expected value of the “up” and “down” values of the put option at the risk-neutral probability and we discount by the risk-free interest rate to get

$$\text{value of put} = 1/1.0168 \times [0.63445 \times \$0 + (1 – 0.63445) \times \$5] = \$1.7975$$

Remember that the value of the put option at the exercise date is Max $(0,X-S_T)$. Review the model solution to Question 9.1 and fill in the no-arbitrage argument.

EUROPEAN CALL OPTION

Exercise 21.16: A stock price is currently \$40. It is known that at the end of three months it will be either \$45 or \$35. The risk-free interest rate with continuous compounding is 8% per annum. Calculate the value of a three-month European call option and of a European put option on the stock with an exercise price of \$40. Solve the problem is the risk-free interest rate is 8% per annum with quarterly compounding.

We use the risk neutral valuation approach, which is fastest for exam problems.

The up movement is 45/40 = 1.125; the down movement is 35/40 = 0.875.

The risk-free interest rate is 8% per annum or 2% per three months.

The risk-neutral probability of an up movement is

$$(e^{2\%} – 0.875) / (1.125 – 0.875) = 0.5808$$

The value of the European call option at the final two nodes is \$5 if the stock rises to \$45 and \$0 if the stock falls to \$35.

The expected value of the call option at the expiration date in a risk-neutral world is

$$0.5808 \times \$5 = \$2.904$$

The present value of the call option is $e^{-4\%} \times \$2.904 = \$2.85$.

The expected value of the put option at the expiration date in a risk-neutral world is

$$(1 – 0.5808) \times \$5 = \$2.096$$

The present value of the put option is $e^{-4\%} \times \$2.096 = \$2.05$.

Quarterly Compounding

If we use quarterly compounding, the three month risk-free interest rate is 1.02 instead of $e^{2\%}$. Everything else in the problem is the same. The risk-neutral probability of an up movement is

$$(1.020 – 0.875) / (1.125 – 0.875) = 0.58$$

The expected value of the call option at the expiration date in a risk-neutral world is

$$0.58 \times \$5 = \$2.9$$

The present value of the call option is $e^{-4\%} \times \$2.9 = \$2.85$.

The expected value of the put option at the expiration date in a risk-neutral world is

$$(1 - 0.58) \times \$5 = \$2.1$$

The present value of the put option is $e^{-4\%} \times \$2.1 = \$2.06$.

Two Period Call and Put

Exercise 21.17: A stock price is currently \$50. Over each of the next three-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a six-month European call option on with a strike price of \$51?

Exercise 21.18: For the situation considered in problem 9.11, what is the value of the six-month European put option with a strike price of \$51? Verify that the European call option and the European put option satisfy put-call parity. If the put option were American, would if ever be optimal to exercise it early at any of the nodes on the tree?

American Options

We discuss the last sentence in this problem. First, not ask the same question about American options in the previous problem?

Answer: American *call* options on non-dividend paying stocks should never be exercised early. If we exercise the American call option early, we lose in two ways. (1) We lose the interest on the cash that we pay early to obtain the underlying asset, and (2) We lose the chance to defer our decision until the last possible date. This latter part, the chance to defer our decision, is particularly important if the current asset price is close to the strike price.

For the put option, if we exercise early, the first of these two pieces is reversed. Instead of losing interest on the cash, we get interest on the cash that we receive early for the sale of the underlying asset. The second part remains the same: we still lose the chance to defer our decision whether or not to exercise the option until the last possible date.

When is it preferable to exercise an American put option at an early date? Answer: When the value of the extra interest exceeds the value of the chance to defer the exercise decision. The value of the interest is easy to determine, since we know the strike price and the risk-free interest rate. The value of deferring the exercise decision depends on various factors and it is hard to estimate. The general rule is as follows: If the put option is deep in the money, and the volatility of the stock price movement is low, and the interest rate is high, then it pays to exercise early. If the put option is near the money, or the volatility of the stock price movement is high, or the risk-free interest rate is low, then it does not pay to exercise early.

This is not a rule to use in solving options pricing problems. This is an intuitive rule to help you understand what is happening in these problems.

European Put Option Valuation

The up movement is +6% and the down movement is –5%. The risk-free interest rate is 5% per annum or 5% × ¼ for a three month period. The value of the risk-neutral probability of an up movement is

$$p = (e^{0.25 \times 5\%} - 0.95) / (1.06 - 0.95) = 0.5689.$$

We determine the values of the stock and the put option at the three nodes in the final column. At the highest node, the value is \$50 × 1.06 × 1.06 = \$56.18; the value of the put option at this node is \$0. At the middle node, the value of the stock is \$50 × 1.06 × 0.95 = \$50.35, and the value of the put option is \$0.65. At the lowest node, the value of the stock is \$50 × 0.95 × 0.95 = \$45.125, and the value of the put option is \$5.875.

The risk-neutral probability of getting to the highest node in the final column is 0.5689^2. The risk-neutral probability of getting to the middle node in the final column is 2 × 0.5689 × 0.4311. The coefficient "2" is included because there are two paths leading to the middle node. The risk-neutral probability of getting to the lowest node in the final column is 0.4311^2.

The present value of the European put option is

$$e^{-5\% \times 0.5} \times (0.5689^2 \times \$0 + 2 \times 0.5689 \times 0.4311 \times \$0.65 + 0.4311^2 \times \$5.875) = 1.376$$

What about the American put option? There are two ways to view this question. One way is to calculate the value of the European put option at the two nodes in the middle column (i.e., at the end of the first period).

☞ At the higher node in the middle column (after 1 period), the stock price is \$50 × 1.06 = \$53. The intrinsic value of the put option is \$0. It is clearly not worth exercising at this node.

☞ At the lower node in the middle column, the stock price is \$50 × 0.95 = \$47.50. The intrinsic value of the put option at this node is \$51.00 – \$47.50 = \$3.50. The value of the put option if we wait one more period is determine by a one period model with an initial price of \$47.50, an up movement of +6%, a down movement of –5%, and a strike price of \$51. This value is \$2.866, which is lower than the intrinsic value of \$3.50. It is better to exercise now than to wait one more period.

☞ At time 0 (the initial node), the intrinsic value of the put option is \$51 – \$50 = \$1. The value of the European put option is \$1.376, as we worked out above. It is not worthwhile to exercise the option at the initial node.

Question: Is there a way to see that we should exercise early at the lower node in the middle column (after 1 period)?

Answer: Yes, the intuition is simple. If we have a put option, and we will exercise the option at both nodes in the next period (both the upward node and the downward node), then we should exercise now and not wait.

Question: What is the intuition for this?

Answer: If we wait, we get the cash from the strike price one period earlier and we can invest this cash at the risk-free interest rate.

Stock Squared

Exercise 21.19: A stock price is currently \$25. It is known that at the end of two months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose that S_T is the stock price at the end of two months. What is the value of a derivative that pays of S_T^2 at this time?

There is nothing special about the square of the stock price. We use the general approach outlined above.

Most of the arithmetic is already done for us in this problem. The up movement is 27/25 = 1.08; the down movement is 23/25 = 0.92 (plus and minus 8%, respectively).

The risk-free interest rate is 10% per annum or 0.1/6 for two months.

The risk-neutral probability of an up movement is $(e^{0.1/6} – 0.92) / (1.08 – 0.92) = 0.60504$.

The value at the two final nodes (at expiration) are $27^2 = 729$ and $23^2 = 529$.

The expected value of the derivative at the expiration date in a risk-neutral world is

$0.60504 \times 729 + (1 – 0.60504) \times 529 = 650.01$

The present value of the derivative is $e^{–0.1/6} \times 650.01 = \$639.26$.

Option Delta

When there is a single period in the binomial tree, the option delta approach is simple.

We purchase one derivative and we short Δ (delta) shares of stock for a net outlay of "f(S) – Δ × S." We choose Δ so that this portfolio is risk-free. This implies that the portfolio has the same value in the "up" state as in the "down" state, or

$$27^2 - \Delta \times 27 = 23^2 - \Delta \times 23$$
$$\text{or } \Delta = 50.$$

We now solve for the present value of the derivative. Since the portfolio is risk-free, it earns a risk-free rate of return over the two months, or

$$[f(S) - 50 \times S\,] \times e^{0.1/6} = 27^2 - 50 \times 27 = 23^2 - 50 \times 23 = -621$$

At time 0, S = \$25. Solving for f(25) gives \$639.26.

EUROPEAN VS AMERICAN

Exercise 21.20: A stock price is current $40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

A. What is the value of a six-month European put option with a strike price of $42?
B. What is the value of a six-month American put option with a strike price of $42?

Immediate Exercise of Options

Exercise 21.21: Estimate how high the strike price has to be in Problem 9.15 for it to be optimal to exercise the option immediately.

RISK-NEUTRAL WORLD

Exercise 21.22: The risk free rate of interest is 5% per annum, the common stock of the ABC Company has a market beta of 1.00, and the market risk premium is 8 points. A May 2001 European call option on ABC stock with an exercise price of $50 has a value of $10 per option on March 1, 2001. If all investors suddenly became risk-neutral, which of the following statements is true?

A. The value of the call option would increase.
B. The value of the call option would remain the same.
C. The value of the call option would decrease.
D. The question can not be answered from the information given.

Risk

In truth, there is no such risk-neutral world; this is only a construct that helps us conceive of option pricing.

> *In a world where investors are risk neutral, the expected return on all securities is the risk-free rate of interest, r.*
>
> *When we move from a risk-neutral world to a risk-averse world, two things happen. The expected growth rate in the stock price changes and the discount rate that must be used for any payoffs from the derivative changes. It happens that these two effects always offset each other exactly.*

Actually a third thing happens as well: The stock price changes at the moment that we turn to a risk-neutral world. That is because the stock is not a real asset, such as a loaf of bread, that has a consumption value to the owner. The stock is a financial security that gives the owner the future cash flows.

Hull is assuming that we change to a risk-neutral world and that the stock price doesn't change. Some readers ask themselves: "Does this makes sense?" Well, Hull is not claiming that it makes sense. In truth, there is no risk-neutral world; this is all just a mathematical construct to help our understanding of option pricing. Instead, just assume that we move to a risk-neutral world and nothing else changes. All current prices stay exactly where they are for the primary securities.

European Option Pricing

EUROPEAN CALL OPTION

Exercise 21.23: A stock trades for $120. Each year, it will rise by 10% or fall by 20%. The risk-free interest rate is 6% per annum. A call option on this stock has an exercise price of $130.

A. What is the risk-neutral probability of an upward movement in the stock price?

B. What is the delta of this option?

C. What should you hold to form a risk-free portfolio?

D. What is the price of a European call option that expires in one year, using the binomial tree pricing method?

E. What would the price of the call option be if it expired in two years (not one)?

F. What would be price of the call option be if the strike price were $135 and it expired in one year?

ONE PERIOD BINOMIAL TREE

(A) The problems doesn't say if the risk-free interest rate is a continuously compounded rate or an effective annual rate. Texts on option pricing generally assume that the risk-free interest rate is continuously compounded. Observed risk-free interest rate are the rates on Treasury securities. These are semiannually compounded if they are rates on Treasury notes or bonds, and they are discount rates if they are rates on Treasury bills.

Question: Does the compounding frequency have a large effect on the price of options?

Answer: Generally the effect is not material. To illustrate this, let us first assume that the rate given in this practice problem is an effective annual rate. Let r = 6% per annum. The risk-neutral probability of a rise in the stock price is

$$[r - (-0.20)] / [+0.10 - (-0.20)] = 0.26 / 0.30 = 0.8667 = 86.67\%.$$

We could also express this as

$$[1 + r - 0.80)] / [1.10 - (0.80)] = 0.26 / 0.30 = 0.8667 = 86.67\%.$$

Intuition: The return from the stock equals the risk-free rate in a risk-neutral world. Let Z be the risk-neutral probability:

$$Z \times 1.10 + (1 - Z) \times 0.80 = 1.06$$
$$(1.10 - 0.80) \times Z + 0.80 = 1.06$$
$$Z = (1.06 - 0.80) / (1.10 - 0.80)$$

We have used annual compounding only to show the method needed if annual compounding is specified in the exam problem. On the exam itself, use continuous compounding unless specifically told to use another compounding frequency; see below.

OPTION DELTA

(B) The delta of the option is the change in the option price divided by the change in the price of the underlying security. When the stock increases by 10%, it is worth \$120 × 1.10 = \$132 and the call option is worth \$2. When the stock decreases by 20%, it is worth \$120 × 0.80 = \$96 and the call option is worth \$0. The change in the option price divided by the change in the price of the underlying security is [\$2 – \$0] / [\$132 – \$96] = \$2/\$36 = 1/18 = 5.56%.

Question: With such a small delta, the stock is probably well out-of-the-money; is that correct?

Answer: Yes; the call option is not worth much. If you calculate a high delta and a low call option price or a low delta and a high call option price, check your work.

(C) To hold a risk-free portfolio, we can purchase 18 options and sell short one share of stock. Alternatively, we can purchase one share of stock and sell short 18 call options.

RISK-NEUTRAL PROBABILITY

(D) The value of the call option in the up state is \$2, and the risk-neutral probability of moving up is 86.67%. The value of the call option in the down state is \$0, so it is not relevant.

The expected value of the call option at the expiration date *in the risk-neutral world* is 86.67% × \$2 = \$1.7333. The expected value of the call option at inception in the risk-neutral world is \$1.7333 / 1.06 = \$1.63522 ≈ \$1.64. Since the value of the call option does not depend on the risk aversion of the investor(s), this is also the value of the call option in the real world.

(E) If the option expires in two years and all other input parameters remain the same, only a few items change in the pricing method.

There are three possible states at the expiration date: up-up, up-down (= down-up), and down-down. The values of the stock and of the call option in these three states are

- up-up: stock = \$120 × 1.10 × 1.10 = \$145.20; call option = \$15.20
- up-down: stock = \$120 × 1.10 × 0.80 = \$105.60; call option = \$ 0.00
- down-down: stock = \$120 × 0.80 × 0.80 = \$ 76.80; call option = \$ 0.00

The risk-neutral probability of two successive up movements is 0.8667 × 0.8667 = 0.7512. This is the only probability that is needed to solve the problem.

The expected value of the call option at the expiration date in the risk-neutral world is 0.7512 × \$15.20 = \$11.41824. The expected value of the call option at the inception date in the risk-neutral world is 0.7512 × \$15.20 = \$11.41824 / $(1.06)^2$ = \$10.16. This is also the value in the real world.

(F) If the strike price is \$135 and there is one year to expiration, the value of the call option is \$0, since the stock price does not exceed the strike price even in the up state.

Question: Is the call option actually worth \$0?

Answer: No. A call option is always worth something (as long as the stock price is not worth \$0), since the stock price might always increase. A one-period binomial tree is a crude approximation, and it gives a price of \$0 for any strike price exceeding \$132.

Question: Doesn't the price determined from the binomial tree pricing method converge to the true option price?

Answer: It does converge, but the convergence is slow. It might take several hundred periods before the calculated option price was exact (to the penny).

Question: Along the path to convergence, does the binomial tree pricing method overstate or understate the true price?

Answer: In most cases, the binomial tree pricing method alternates. Suppose the true price of the option is $10. The binomial tree pricing method with one valuation period may show a price of $9; with two periods, it may give a price of $11; with three periods, it may give a price of $9.50; and so forth.

Question: If the interest rate is continuously compounded, how does the solution change?

Answer: The changes are minor.

- For the risk-neutral probability we use $r = e^{0.06}$ instead of 1.06.
- The continuously compounded interest rate is 1.061837 – 1 = 6.1837%.
- The risk-neutral probability is [1.061837 – 0.80] / 0.30 = 0.87279.
- We discount from the expiration date to the inception date by 1.061837, not by 1.06.

The value of the call option at the inception date is 0.87279 × $2 / 1.061837 = $1.6439 ≈ $1.64. The value of the call option does not change much for a slight change in the risk-free interest rate.

Question: The value of the call option increased when the risk-free interest rate increased. Is this always true?

Answer: Yes, this is always true. The call option allows the investor to hold onto the cash until the expiration date, instead of using the cash to purchase the stock at the inception date. The greater the risk-free interest rate, the more valuable is the ability to hold onto the cash, since the investment income on the cash is greater.

Question: It seems like there is a second reason as well. If the risk-free interest rate is greater, then the expected value of the stock at the expiration date in the risk-neutral world is greater.

Answer: Not quite. If the risk-free interest rate is greater, then the discount rate in the risk-neutral world from the expiration date to the inception date is also greater. This exactly offsets the expected increase in the value of the stock in the risk-neutral world.

Question: Can one see the effect of the higher interest rate in the binomial tree pricing method?

Answer: Yes. Let R be the original risk-free interest rate and let R’ be the higher risk-free interest rate. The original risk-neutral probability is (R – D)/(U – D), and the revised risk-neutral probability is (R’ – D)/(U – D). To go from the value of the call option at the *expiration* date in the risk-neutral world using the original interest rate of R to the revised interest rate of R', we multiply by (R' – D)/(R – D).

In addition, we discount by R' instead of by R. The multiplicative factor to go from using the original interest rate of R to the revised interest rate of R' is R/R'.

To go from the value of the call option at the *inception* date in the risk-neutral world using the original interest rate of R to the revised interest rate of R', we multiply by the product of these two factors, or (R' – D)/(R – D) × R/R'. As long as D > 0, R > D, and R' > D, this product is greater than unity.

RETURNS

Exercise 21.24: A stock has a current price of \$80 and a market beta of 1.2 and a volatility of 32%. The risk-free interest rate is 7% per annum and the market risk premium is 8%. All rates use continuous compounding.

A. What is the expected value of the stock in six months?
B. What is the expected rate of return that is realized over this half year period?

Return vs Rate of Return

Part (a): The CAPM formula uses effective annual rates of return. The expected return on this stock using annual effective rates of return is

$$(e^{0.07} - 1) + 1.2 \times (e^{0.08} - 1) = 0.172453.$$

The continuously compounded rate of return on this stock = *ln*(1.172453) = 15.9% per annum. The expected value of the stock in six months \$80 × $e^{0.159 * ½}$ = \$86.62.

Part (b): The expected rate of return that is realized over this half year period depends on the probability distribution of the stock price. We assume that stock prices have a lognormal distribution, so the rates of return have a normal distribution. The expected (annual) rate of return that is realized over any period is $\mu - \sigma^2/2$ = 10.8%.

Question: Should we divide 10.8% in half for a 5.4% continuously compounded rate of return?

Answer: This depends on the examiner. Some examiners want the annual rate of return; others want the rate of return over a half year period.

Question: Why does the CAPM use annual effect rates of return whereas option pricing texts use continuously compounded rates of return?

Answer: The simple answer is that the derivation of the CAPM is based on annual effective rates of return. Investment theorists differ on the proper rates for return factor models.

European Calls and Puts

Exercise 21.25: A stock's current price is \$110. The expected rate of return is 15% per annum, the volatility is 25% per annum, and the risk-free interest rate is 11% per annum. A European call option and a European put option on this stock have 43 days to expiration.

A. What is the price of the European call option if the strike price is \$105?
B. What is the price of the European put option if the strike price is \$140?

Black-Scholes

Part (a): We use the Black-Scholes model to solve for the prices of European options. The input data to the Black-Scholes formula are

- S_0 = \$110
- K = \$105
- r = 11% per annum
- σ = 25% per annum
- t = 43/365 = 0.117808

We begin by calculating values for d_1 and d_2. The formulas are:

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma(\sqrt{T-t})}$$

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma(\sqrt{T-t})} = d_1 - \sigma\sqrt{(T-t)}$$

The values of d_1 and d_2 in this problem are

$$d_1 = \{ \ln(110/105) + [0.11 + ½ (0.25)^2](43/365) \} / \{ 0.25 \times (43/365)^{0.5} \} = 0.7361.$$

$$d_2 = d_1 - \sigma \times \sqrt{t} = 0.7361 - 0.25 \times (43/365)^{0.5} = 0.6503.$$

We use a cumulative normal distribution table or a built-in spreadsheet function to calculate

$$N(d_1) = 0.769165 \text{ and } N(d_2) = 0.742251.$$

The values of the put and call options in the Black-Scholes formula are

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

$$p = Xe^{-r(T-t)}N(-d_2) - SN(-d_1)$$

The value of the European call option is

$$\$100 \times 0.769165 - \$105 \times e^{-0.11 \times 43/365} \times 0.742251 = \$7.6752.$$

Part (b): The put option is computed in a similar fashion. The only input figure which changes is the strike price, which is now \$140. For the put option, the values of d_1 and d_2 are

$$d_1 = \{ \ln (110/140) + [0.11 + ½ (0.25)^2](43/365) \} / \{ 0.25 \times (43/365)^{0.5} \} = –2.6177.$$

$$d_2 = d_1 – \sigma \times \sqrt{t} = –2.6177 – 0.25 \times (43/365)^{0.5} = –2.7035.$$

We use a cumulative normal distribution table or a built-in spreadsheet function to calculate

$$N(–d_1) = 0.995574 \text{ and } N(–d_2) = 0.996569.$$

The value of the European put option is

$$\$140 \times e^{–0.11 \times (43/365)} \times 0.996569 – \$110 \times 0.995574 = \$28.21.$$

Dividends

Exercise 21.26: A stock trading for $75 has an expected return of 16% per annum and a volatility of 35% per annum. The risk-free rate is 9% per annum. All rates use continuous compounding.

1. If the stock pays no dividends, find the prices of a European call option and European put option that expire in 150 days and that have a strike price of $70.

2. If the stock pays a continuous dividend of 4% per annum, find the prices of a European call option and European put option that expire in 150 days with a strike price of $70.

No Dividends

Part (a): If there are no dividends, the computation is as follows:

The input data to the Black-Scholes formula are

- S_0 = \$75
- K = \$70
- r = 9% per annum
- σ = 35% per annum
- t = 150/365 = 0.411

We calculate values for d_1 and d_2.

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma(\sqrt{T-t})}$$

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma(\sqrt{T-t})} = d_1 - \sigma\sqrt{(T-t)}$$

The values of d_1 and d_2 are

$$d_1 = \{ \ln(75/70) + [0.09 + ½ (0.35)^2](150/365) \} / \{ 0.35 \times (150/365)^{0.5} \} = 0.5845$$

$$d_2 = d_1 – \sigma \times \sqrt{t} = 0.5845 – 0.35 \times (150/365)^{0.5} = 0.3601.$$

We use a cumulative normal distribution table or a built-in spreadsheet function to calculate

$$N(d_1) = 0.720558 \text{ and } N(d_2) = 0.640614.$$

Similarly,

$$N(–d_1) = 1 – 0.720558 = 0.279442 \text{ and } N(–d_2) = 1 – 0.640614 = 0359386.$$

The values of the put and call options in the Black-Scholes formula are

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

$$p = Xe^{-r(T-t)}N(-d_2) - SN(-d_1)$$

The value of the European call option is

$$\$75 \times 0.7205587 - \$55 \times e^{-0.09 \times 150/365} \times 0.640614 = \$54.04 - \$43.21 = \$10.83.$$

The put option is computed in a similar fashion. We compute the values of $N(-d_1)$ and $N(-d_2)$:

$$N(-d_1) = 0.279442 \text{ and } N(-d_2) = 0.359386.$$

The value of the European put option is

$$\$70 \times e^{-0.09 \times (150/365)} \times 0.359386 - \$75 \times 0.279442 = \$24.24 - \$20.96 = \$3.29.$$

Dividends

Part (b): The continuous dividend is interpreted as an adjustment to the stock price. Instead of buying one share of stock for S_0 at time 0, we purchase an amount of stock equal to $S_0 \times e^{-r\theta}$ at time 0 and we reinvest the continuous dividends in additional shares of stock. At time
T, the maturity date of the option, we have one share of stock S.

The revised stock price at time 0 is $\$75 \times e^{-0.04 \times 150/365}$. The Black-Scholes expressions for d_1 and d_2 for the call option are

$$d_1 = \{ \ln (75 \times e^{-0.04 \times 150/365}/70) + [0.09 + ½ (0.35)^2](150/365) \} / \{ 0.35 \times (150/365)^{0.5} \} = 0.5113$$

$$d_2 = d_1 - \sigma \times \sqrt{t} = 0.5113 - 0.35 \times (150/365)^{0.5} = 0.2869.$$

We can rewrite the equation for d_1 as

$$d_1 = \{ \ln (75/70) + [0.09 - 0.04 + ½ (0.35)^2](150/365) \} / \{ 0.35 \times (150/365)^{0.5} \} = 0.5113$$

This is the more common method of valuing d_1 when there is a continuous dividend payment.

We evaluate the cumulative normal distribution values as

$$N(d_1 = 0.5113) = 0.695430 \text{ and } N(d_2 = 0.2869) = 0.612906$$

and

$$N(-d_1 = -0.5113) = 0.304570 \text{ and } N(-d_2 = -0.2869) = 0.387904$$

The value of the European call option is

$$\$75 \times e^{-0.04 \times 150/365} \times 0.695430 - \$55 \times e^{-0.09 \times 150/365} \times 0.612906 = \$9.96.$$

The value of the European put option is

$$\$70 \times e^{-0.09 \times (150/365)} \times 0.387094 - \$75 \times e^{-0.04 \times 150/365} \times 0.304570 = \$3.64.$$

American Options

AMERICAN PUTS AND CALLS

Exercise 21.27: We are using a binomial lattice to price an American call option and an American put option on a stock that pays no dividends. The current stock price is \$120 and the exercise price for both the put and the call is \$110. The volatility of the stock returns is 40% per annum, and the risk-free interest rate is 10% per annum. The options expire in 120 days.

1. When (if ever) should each option be exercised?
2. What is the value of a European call option?
3. What is the value of the American call option?
4. What is the value of a European put option?
5. What is the value of the American put option?

The options expire in 120 days. The four period binomial tree has 30 days in each period.

The volatility of the stock price is 40% per annum. The upward movement in the binomial tree is $e^{\sigma\sqrt{\Delta t}} = e^{0.40 \times \sqrt{(30/365)}}$.

$$0.40 \times \sqrt{(30/365)} = 0.114676, \text{ and } e^{0.40 \times \sqrt{(30/365)}} = e^{0.114676} = 1.1215.$$

The downward movement is the reciprocal of the upward movement, or 1/1.1215 = 0.8917.

The risk-neutral probability of an upward movement equals

$$(e^{r \times \Delta t} - D) / (U - D) = (e^{0.1 \times 30/365} - 0.8917) / (1.1215 - 0.8917) = 0.5073.$$

The risk-neutral probability of a downward movement is the complement of the risk-neutral probability of an upward movement, or 1 – 0.5073 = 0.4927.

The risk-free discount rate for each period is $e^{-r \times \Delta t} = 0.9918$.

STOCK PRICE BINOMIAL TREE

We draw the binomial tree for the stock price, with an upward movement of 1.1215 and a downward movement of 0.8917. Since the stock does not pay dividends, the American call option has the same value as the European call option. However, the American put option may be more valuable than the European call option. We show the valuation procedure for both types of option, since the subsequent problems deal with dividend paying stocks.

Figures BL.1: Stock Price Binomial Lattice

								189.8430
						169.2744	<	
				150.9343	<			150.9343
		134.5813	<			134.5813	<	
120.000	<			120.0000	<			120.0000
		106.9986	<			106.9986	<	
				95.4058	<			94.4058
						85.0690	<	
								75.8521
time = 0		time = 1		time = 2		time = 3		time = 4

Call Option Binomial Tree

We draw the binomial tree for the call option, using the strike price of $110 and the binomial tree for the stock price. The figure below shows two prices for each cell. The upper price is the price of the corresponding European call option, based on the binomial tree pricing method. The lower value is the intrinsic value of the option. If the intrinsic value exceeds the value of the European call option, we should exercise the American call option early.

								79.84
						60.17 59.27	<	
				42.73 40.93	<			40.93
		28.96 24.58	<			25.48 24.58	<	
18.93	<			15.28 10.00	<			10.00
		8.92 0.00	<			5.03 0.00	<	
				2.53 0.00	<			0.00
						0.00 0.00	<	
								0.00
time = 0		time = 1		time = 2		time = 3		time = 4

We determine the value of the option in the last column from the stock price and the strike price. We use a backwards recursive method to determine the value of the option in each preceding column.

- To determine the stock prices, we move from left to right.
- To determine the option prices, we move from right to left.

Illustration A: The value in the uppermost node in the right-most column of 79.84 equals the stock price minus the strike price, or 189.843 – 110.000 = 79.84 (rounded).

Illustration B: The topmost cell in the preceding column has two values: 60.17 and 59.27. The upper value is the value of the European call option in that cell. The lower value is the intrinsic value of the call option in that cell, which is the value that the long position receives if the call option is exercised early.

- The value of 60.17 is *(0.5073 × 79.84 + 0.4927 × 40.92) × 0.9918 = 60.17*
- The value of 59.27 is the stock price minus the strike price = 169.2744 – 110.000 = 59.27.

In every cell, the European call option price exceeds the intrinsic value, so the American call option has the same value as the European call option. This is always true for options on non-dividend paying stocks.

PUT OPTION BINOMIAL TREE

We draw the binomial tree for the put option, using the strike price of $110 and the binomial tree for the stock price. Each cell has two prices. The upper figure is the value of the corresponding European call option, based on the binomial tree pricing method. The lower figure is the intrinsic value of the option. If the intrinsic value exceeds the value of the European put option, we should exercise the American put option early.

								0.00
						0.00 0.00	<	
				0.00 0.00	<			0.00
		1.70 0.00	<			0.00 0.00	<	
5.48	<			3.49 0.00	<			0.00
		9.46 3.00	<			7.13 3.00	<	
				15.77 14.59	<			14.59
						24.03 **24.93**	<	
								34.15
time = 0		time = 1		time = 2		time = 3		time = 4

When the intrinsic value of the put option exceeds the value of the corresponding European put option, the value of the corresponding European put option is replaced by the intrinsic value and the figures are in boldface. *The recursive valuation procedure uses the higher of the intrinsic value and the European put option value to determine the value of the American put option in the preceding cells.*

Intuition

Question: How can we tell whether it would be optimal to exercise an American put option early? Is there any simple rule-of-thumb?

Answer: For the binomial tree pricing method, there is a rule-of-thumb that is sufficient, though it is not necessary. If the cell in the binomial tree satisfies this rule-of-thumb, it is optimal to exercise the American put option early. However, it may be optimal to exercise early even when the rule-of-thumb is not satisfied.

There are offsetting forces for the early exercise of an American put option:

1. By exercising early, the investor loses the volatility value of holding the option. The stock value may decline further, increasing the value of the put option. The stock value may rise, but this downside risk is capped at zero.
2. By exercising early, the investor gains the investment income on the strike price. This gain equals the strike price × (e^{rt} – 1), where "r" is the risk-free interest rate and "t" is the remaining time to maturity. The extra investment income must be discounted to the current date (the cell date):

$$\text{strike price} \times (e^{rt} - 1) \times (e^{-rt}) = \text{strike price} \times (1 - e^{-rt})$$

Consider the lowest cell in the column labeled "time=4" in the binomial tree above. The intrinsic value in this cell is $24.93. Both paths from this cell lead to cells in the fifth column which have positive payoffs for the put option. The intrinsic values in these two cells in the fifth column (bottom two nodes) are $14.59 and $34.15.

Since the investor in the bottom cell of the fourth column will exercise the American put option in the fifth column no matter which path is taken, the volatility value of waiting is zero. The value of the investment income is positive, so it is advantageous to exercise early.

Question: Can you generalize these comments?

Answer: The general rule is as follows. When the option is at the money (that is, when the stock price equals the strike price), the value of volatility is greatest. (To be exact, we should define "at the money" to mean that the stock price equals the present value of the strike price.)

When the option is deep in the money or deep out of the money, the value of volatility is low. When the investment income exceeds the value of volatility, it pays to exercise the option early.

When the option is deep out of the money, it has no intrinsic value, so we do not exercise early. When the option is deep enough in the money that the value of volatility is less than the expected investment income from early exercise, it pays to exercise early,

The values in the bottom node in column four are shown in bold. In this cell, it is advantageous to exercise the American put option early. The lower value of $24.93 (the intrinsic value) replaces the European put option that was originally shown as the upper value. The original value was

$$(0.5073 \times 14.59 + 0.4927 \times 34.15) \times 0.9918 = 24.03$$

To check the rule-of-thumb, we examine the present value of the investment income on the strike price. The strike price is \$110 and the value of $(1 – e^{-rt})$ for one period is 0.0082. The product of \$110 and 0.0082 is \$0.90, which is the difference between \$24.93 and \$24.03. The additional value from early exercise is the value of the additional investment income.

AMERICAN OPTIONS WITH DIVIDENDS

Exercise 21.28: A stock trading for $120 has a volatility of 40% per annum. The risk-free rate is 10% per annum. American put and call options expire in 120 days with an exercise price of $110.

A. Assume that the stock pays no dividends. Using a four-period binomial tree pricing method, draw the stock price tree and the trees for the call and the put. Explain when, if ever, each option should be exercised. What is the value of a European call option? What is the value of the American call option? What is the value of a European put option? What is the value of the American put option?

B. Assume that the stock will pay a dividend of 3% of its value in 50 days. Using a four-period binomial tree pricing method, draw the stock price tree and the trees for the call and the put. Explain when, if ever, each option should be exercised. What is the value of a European call option? What is the value of the American call option? What is the value of a European put option? What is the value of the American put option?

C. Assume that the stock will pay a dividend of $3 in 50 days. Using a four-period binomial tree pricing method, draw the stock price tree and the trees for the call and the put. Explain when, if ever, each option should be exercised. What is the value of a European call option? What is the value of the American call option? What is the value of a European put option? What is the value of the American put option?

MORE DIVIDENDS

Exercise 21.29: A stock currently trades for $80 with an expected return of 15% per annum and a volatility of 30% per annum. The stock is expected to pay a $5 dividend in 70 days. The risk-free interest rate is 10% per annum. Options written on this stock have an exercise price of $80 and expire in 120 days. Calculate the price of these options using a four period tree.

A. Draw the stock price tree.
B. Calculate the values of European and American call and put options written on this stock. Value the options using the recursive procedure. Construct the price trees for each option.
C. Compare the prices of the European and American options. How much value does the right to exercise the option before expiration add to the value of the American options?
D. Explain when, if ever, each option should be exercised.

PRELIMINARIES

We show just the preliminaries; the binomial tree pricing method is the same as in the previous practice problem.

The options expire in 120 days. The four period binomial tree has 30 days in each period.

The volatility of the stock price is 30% per annum. The "up" movement in the binomial tree is $e^{\sigma\sqrt{\Delta t}} = e^{0.30 \times \sqrt{(30/365)}}$.

$$0.30 \times \sqrt{(30/365)} = 0.086007, \text{ and } e^{0.30 \times \sqrt{(30/365)}} = e^{0.086007} = 1.089814.$$

The "down" movement is the reciprocal of the up movement, or 1/1.089814 = 0.9176.

The risk-neutral probability of an up movement equals

$$(e^{r \times \Delta t} - D) / (U - D) = (e^{0.1 \times 30/365} - 0.9176) / (1.0898 - 0.9176) = 0.5264.$$

The risk-neutral probability of a down movement is the complement of the risk-neutral probability of an up movement, or 1 – 0.5264 = 0.4736.

The risk-free discount rate for each period is $e^{-r \times \Delta t} = 0.9918$.

STOCK PRICE BINOMIAL TREE

The manner of drawing the binomial tree for the stock price differs when the stock pays dividends. For a fixed dollar dividend, we subtract the present value of the dividend from the initial stock price, draw the tree, and then add back in the appropriate dividend value to the cells preceding the ex-dividend date.

The present value of the dividend at time 0 is $\$5.00 \times e^{0.01 \times (70.365)} = \$4.9050$. The adjusted stock price at time 0 is $\$80.0000 - \$4.9050 = \$75.0950$.

CAS Spring 2003, Exam 8, Question 36 (2.5 points)

A stock price is currently \$40. It is known that at the end of one month the stock's price will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding.

1. (1 point) Determine the value of a one-month European call option with a strike price of \$39.

2. (1.5 points) Assume that the expected return on the stock is 10% as opposed to the risk-free rate. What is the correct discount to be applied to the payoff in the real world?

Show all work.

Part A: The upward and downward movements in the stock price are ± \$2/\$40 = ± 5%, or 1.05 and 0.95. The risk-neutral probability of an upward movement in the stock price is

$$(e^{0.08 / 12} - 0.95) / (1.05 - 0.95) = 56.69\%.$$

If the stock price moves up, the option payoff is \$3; if the stock price moves down, the option payoff is zero. The expected option payoff in a risk-neutral world is 56.69% × \$3 = \$1.70. The present value of this expected payoff at the risk-free interest rate is \$1.70 × $e^{-0.08 / 12}$ = \$1.69.

Part B: In the real world, the probability of the stock price moving up is

$$(e^{0.10 / 12} - 0.95) / (1.05 - 0.95) = 58.37\%.$$

The expected option payoff is 58.37% × \$3 = \$1.75. The capitalization rate for the call option is (\$1.75 / \$1.69 – 1) × 12 = 42.60%. Since we used only two decimal places for the dollar amounts, the answer is not exact; for the exam, use fractions of a penny for the dollar amounts so that all the digits in the answer are significant.

CAS Spring 2003, Exam 8, Question 39 (3 points)

- The current price of a stock is $50.
- The stock value either increases by 6% or decreases by 5% every six months.
- The risk-free rate is 3% per annum with continuous compounding.
- Determine the value of a one-year European call option with a strike price of $52.

Show all work.

Solution:

Step 1: The upward movement is 1.06 and the downward movement is 0.95. The risk-free interest rate for a six month period $e^{0.015}$.

Step 2: The risk-neutral probability of an upward movement is

$$(e^{0.015} – 0.95) / (1.06 – 0.95) = 59.19\%.$$

Step 3: The stock price in one year's time may take three values, for which the call option values and the risk-neutral probabilities are as shown below:

Top node: stock = \$50 × 1.06^2 = \$56.18; option = \$4.18; probability = $59.19\%^2$ = 35.03%
Middle node: stock = \$50 × 1.06 × 0.95 = \$50.35; option = \$0; probability = 2 × 59.19% × (1 – 59.19%) = 48.31%
Bottom node: stock = \$50 × 0.95^2 = \$45.13; option = \$0; probability = $(1 – 59.19\%)^2$ = 16.65%

The expected value of the option at the final column of nodes (in one year) is

$$\$4.18 \times 35.03\% = \$1.46.$$

The present value at time 0 is \$1.46 × $e^{–0.03}$ = \$1.42.